DEPARTMENT OF ECONOMICS

Working Paper

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INVESTMENTS UNDER CREDIT CONSTRAINTS

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Working Paper No. 2000-7

ISSN 1396-2426

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A Dynamic Agricultural Household Model with Uncertain Income and Irreversible and Indivisible Investments under Credit Constraints

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June 16, 2000

Abstract

A dynamic model of agricultural household behaviour in less developed countries in the presence of credit constraints and income uncertainty is developed. The production side of the model takes into account the irreversible and indivisible nature of non-stationary agricultural investment options, thereby combining the standard intertemporal consumption model with features from the real option pricing literature. The model framework represents an interesting dynamic alternative to the static household models in the literature. It is shown that the model can be solved by use of dynamic programming routines, and numerical results are obtained for a variety of parameter values. Several interesting results emerge: First, the consumption policy functions of the agricultural household can become highly non-linear, violating the standard result that increased wealth implies increased consumption. This has implications for empirical estimation. Secondly, increasing uncertainty does not in general reduce the average propensity to consume, or increase the propensity to save. Thirdly, increased variation of income only to a limited extent carries through to variation in consumption. Fourthly, the effects of shocks to income depend crucially on the timing of shocks. And finally, reducing uncertainty does not reduce poverty significantly, whereas agricultural extension services and cash transfers are more likely to do so.

Keywords: agriculture, household model, credit constraints, uncertainty, irreversibility, indivisibility, dynamic programming.

JEL: Q12, O12, O13, D1, D9, C61.

Acknowledgements: The authors wish to thank Birgit Grodal, Finn Helles, Rasmus H. Jacobsen, and Jamsheed Shorish for constructive criticism and useful suggestions. Financial support from the Danish Council for Development Research and the Danish Agricultural and Veterinary Research Council is gratefully acknowledged.

1 Introduction

The present paper provides a theoretical framework for modelling the intertemporal behaviour of the agricultural household in the presence of credit constraints and income uncertainty. In particular, the irreversible and indivisible nature of non-stationary agricultural investment options is taken into account. Previously, the theoretical literature has treated the issue of consumption smoothing under credit constraints and income uncertainty separately from the investment issue. However, there is empirical evidence of an interaction between these.

To pick out just one example: Data from a household survey in Nicaragua in 1996, covering approximately 1500 agricultural households, support the widely held belief that peasants in less developed countries are severely restricted in their access to credit.[1] Among the surveyed households, only 10% reported that they had access to credit. With respect to savings, 49% of the households were found to have a positive stock of domestic animals, which are to a large extent used as a substitute for financial savings.[2] Out of these households, 40% sold animals during 1996, and "investment other than land purchase" was advanced as the underlying reason by 42% of the households, whereas 29% of the sales were for consumption purposes. Furthermore, 56% per cent of the households with domestic animals reported a positive accumulation rate of animals during the years 1990-96.

These figures clearly indicate that when farmers are credit constrained, both investment and consumption motives become important determinants of saving behaviour. Savings are used both to finance investments and to smooth consumption. The lack of credit for consumption smoothing purposes is particularly important, considering the often highly uncertain agricultural income and the lack of markets for risk. The severeness of missing loans for investments is amplified through the often indivisible and irreversible nature of agricultural investments. Examples include the investment in a well, which is clearly both indivisible and irreversible, whereas the purchase of other farm equipment is perhaps only irreversible to a certain degree, but obviously still indivisible. Education and training are yet other options which can be thought of as investments which are both fully irreversible and, to some extent, indivisible.

When credit markets fail, the separability between production and consumption decisions breaks down. This has long been recognised in the literature and has caused the emergence of "agricultural household models",

[1] Information about the survey can be found in Davis, Carletto, and Sil (1997).

[2] This is especially the case among the poorer households, which only derive very limited current income from domestic animals due to inefficient breeding methods.

see e.g. Singh, Squire, and Strauss (1986) or Sadoulet and de Janvry (1995). However, most of these models are static models of limited value for understanding the *process* of development, which is indeed an important objective of development economics.

On the other hand, standard intertemporal consumption models with exogenous income and credit constraints, as analysed by e.g. Deaton (1991), though dynamic and perhaps suitable for the case where wage labour is the primary source of income, are not particularly relevant for analysing rural households where income from farm production contributes significantly to total household income. This requires a model which incorporates both production and consumption behaviour.

Recent investment theory, see e.g. Dixit and Pindyck (1994), has emphasised the option nature of investments arising from irreversibility, indivisibility, and uncertainty, together with leeway in the timing of investments. This paper extends the standard intertemporal consumption model with features from this literature to provide a dynamic alternative to the existing static household models.

Two previous studies, Rosenzweig and Wolpin (1993) and Fafchamps and Pender (1997), have attempted to analyse the implications of indivisible and irreversible investments for household behaviour when credit constraints are present. However, the models used are not explicitly dynamic in the sense that the investment decision is a "once and for all" decision with the cost and the quality of the investment being independent of time, thereby eliminating the option value aspect. Furthermore, both papers are aimed at empirical estimations and therefore pay less attention to the theoretical mechanisms at work in the models. Thus, the model in this paper is a theoretical extension of these models to a fully dynamic set-up by incorporating a true value of waiting and allowing for repeated investments.

More specifically, the model in this paper incorporates the following features: First, farmers in less developed countries are imitators rather than innovators. This is captured by an exogenous process of technology growth. Secondly, to capitalise on existing techniques, the farmer has to invest in equipment and/or education. These investments are both indivisible and irreversible, and they must be recurred every time the farmer wishes to increase his state of technology. Thirdly, the farmer is credit constrained, implying that production and consumption decisions must be analysed simultaneously. In order to invest, the farmer must save and thereby give up or postpone consumption. Finally, income is uncertain and risk sharing devices are absent, thereby creating a motive for precautionary saving.

It is shown that the model can be solved by use of dynamic programming routines, and numerical results are obtained for a variety of parameter choices. Several interesting results emerge. First, the policy functions of the agricultural household can become highly non-linear, violating the standard result that increased wealth implies increased consumption. This has important implications for empirical estimation. Secondly, increasing uncertainty does not in general reduce the average propensity to consume, or increase the propensity to save, and it might cause increased investment activity. Thirdly, increased variation of income only to a limited extent carries through to variation in consumption. Fourthly, the effects of shocks to income depend crucially on the timing of shocks. Finally, reducing uncertainty does not reduce poverty significantly, whereas agricultural extension services and cash transfers are more likely to do so.

In Section 2, a deterministic version of the model is presented. The results of the numerical simulations are presented and interpreted in Section 3. Section 4 introduces uncertainty into the model, and the numerical simulations for the stochastic version of the model are contained in Section 5. Section 6 contains a discussion of the model and its properties, and compares it to related models in the literature. This section also contains directions for future research. Finally, Section 7 concludes the paper. The Appendix contains proofs of the propositions in the paper.

2 The Deterministic Model

This section presents the deterministic version of the household model. The consumption side of the model resembles very closely the standard set-up from the dynamic consumption models with buffer-stock saving, see e.g. Miller (1974), Mendelson and Amihud (1982), Zeldes (1989), Deaton (1991), and Carroll (1997). However, the standard consumption model is in this paper extended with a production side inspired by the optimal-stopping literature, see e.g. Dixit and Pindyck (1994).

Specifically, the farmer maximises infinite horizon utility given by the following time separable utility function:

$$U = \sum_{t=0}^{\infty} (1+\delta)^{-t} u(c_{t+1})$$

where $\delta > 0$ is the subjective rate of discount. Throughout the paper, it will be assumed that the current-period utility function, u, is given by:

$$u(c_{t+1}) = \begin{cases} \frac{1}{1-\eta} c_{t+1}^{1-\eta} & , \quad \eta > 0 \quad \wedge \quad \eta \neq 1 \\ \log(c_{t+1}) & , \quad \eta = 1 \end{cases}$$

For $\eta \geq 1$, it is assumed that $u(0) = \lim_{c_t \to 0} u(c_t) = -\infty$ and $u'(0) = \lim_{c_t \to 0} u'(c_t) = +\infty$.

Production by the farmer in period t depends solely on his current level of installed technology, Θ_t. To simplify, it will be assumed that the production function exhibits constant returns to scale in Θ_t:

$$y_t = F(\Theta_t) = y\Theta_t$$

where y is some positive constant. The farmer has the option to increase his level of technology by investing in the current level of exogenous technology in the economy, θ_t, which grows at the rate of $\alpha > 0$:

$$\theta_t = (1+\alpha)^t \theta_0$$

The timing of the problem is the following: The farmer enters period t with a given level of wealth (or cash) and with installed technology Θ_{t-1}. Wealth is then divided between consumption, investment, and savings. The investment decision is a discrete choice between investing in the currently available level of the exogenous technology, θ_{t-1}, and refraining from investing altogether. In case the latter is chosen, the level of installed technology remains unchanged, i.e. $\Theta_t = \Theta_{t-1}$. However, if an investment is undertaken, then $\Theta_t = \theta_{t-1}$. The cost of investment is given by:

$$\omega(\Theta_t, \Theta_{t-1}) = \begin{cases} \omega_1\theta_{t-1} + \omega_2(\theta_{t-1} - \Theta_{t-1}) & if \quad \Theta_t = \theta_{t-1} \\ 0 & if \quad \Theta_t = \Theta_{t-1} \end{cases}$$

where ω_1 are ω_2 are non-negative constants. A positive value of ω_1 implies that the investment contains an indivisible element, whereas a positive value of ω_2 implies the existence of an adoption cost which is increasing in the size of the technology change. Income, y_t, is realised at the end of period t and together with savings, s_t, and the exogenous rate of interest, r, it determines the amount of wealth, $(1+r)\,s_t + y\Theta_t$, to be carried into the following period. The timing of events is summarised in Figure 1.

The credit constraint implies that savings must be non-negative in all periods:

$$s_t \geq 0 \quad , \quad t \geq 0$$

Furthermore, it is assumed throughout that $\delta \geq r > \alpha$. The technical importance of these assumptions will become clearer in the following sections. However, the assumption of a relatively impatient agent, $\delta \geq r$, is a fairly standard assumption in the literature and can be found in Deaton (1990), Deaton (1991), Carroll (1997), and Fafchamps and Pender (1997).

The above set-up is formalised, and properties of the model and the solution approach are analysed in the following subsections. The numerical simulations of the model are presented in Section 3.

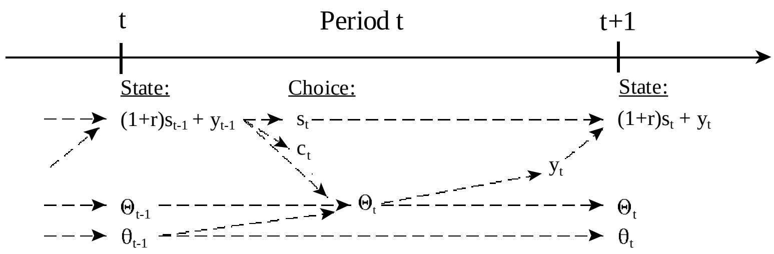

Figure 1: Timing of events

2.1 The Optimisation Problem

To simplify notation, it is instructive to define a vector of variables: $z_t = (s_t, \Theta_t, \theta_t)$ for $t \geq 0$. The optimisation problem of the farmer can then be stated formally as a sequence problem:

$$\begin{array}{ll} \sup\limits_{\{z_{t+1}\}_{t=0}^{\infty}} & \sum\limits_{t=0}^{\infty} (1+\delta)^{-t} u(z_t, z_{t+1}) \\ \text{st} & z_{t+1} \in \Gamma(z_t), \quad t = 0, 1, 2, ... \\ & z_0 \in \mathbb{Z} \end{array} \tag{1}$$

where $\mathbb{Z}$ is some subset of $\mathbb{R}_+^3$ to be characterised later.[3] Since $c_{t+1} = (1+r)s_t + y\Theta_t - s_{t+1} - \omega(\Theta_{t+1}, \Theta_t)$, current-period utility takes the form:

$$u(z_t, z_{t+1}) = \begin{cases} \frac{1}{1-\eta}\left((1+r)s_t + y\Theta_t - s_{t+1} - \omega(\Theta_{t+1}, \Theta_t)\right)^{1-\eta}, & \eta > 0 \wedge \eta \neq 1 \\ \log\left((1+r)s_t + y\Theta_t - s_{t+1} - \omega(\Theta_{t+1}, \Theta_t)\right), & \eta = 1 \end{cases} \tag{2}$$

The correspondence, $\Gamma : \mathbb{Z} \rightarrow \mathbb{Z}$, describing the feasible set is given by:

$$\Gamma(z_t) = \left\{ z_{t+1} \in \mathbb{Z} : \begin{array}{l} \Theta_{t+1} \in \{\Theta_t, \theta_t\} \\ s_{t+1} + \omega(\Theta_{t+1}, \Theta_t) \leq (1+r)s_t + y\Theta_t \\ \theta_{t+1} = (1+\alpha)\theta_t \end{array} \right\} \tag{3}$$

The constantly growing technology creates an unboundedness of the problem in (1)-(3) which complicates the analysis. In order to be able to characterise the solution to the problem more specifically and to apply dynamic programming techniques, it is instructive to first impose a normalisation on the problem.

[3] $\mathbb{R}_+^l$ will be taken to be the subspace of $\mathbb{R}^l$ containing all non-negative vectors of size l.

2.2 The Normalised Problem

The problem in (1)-(3) is normalised by considering $(1+\alpha)^{-t} z_t$ instead of z_t. This implies that the normalised state of exogenous technology becomes constant, $\tilde{\theta}_t = (1+\alpha)^{-t}\theta_t = \theta_0$. It can therefore be treated as a parameter, leaving only two variables: $\tilde{s}_t$ and $\tilde{\Theta}_t$.

Thus, define $\tilde{z}_t = (\tilde{s}_t, \tilde{\Theta}_t)$ for $t \geq 0$. The sequence problem in (1)-(3) can then be rewritten in terms of $\tilde{z}_t$:

$$\begin{aligned} \sup_{\{\tilde{z}_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t \tilde{u}\left(\tilde{z}_t, \tilde{z}_{t+1}\right) \\ \text{st} \quad & \tilde{z}_{t+1} \in \tilde{\Gamma}\left(\tilde{z}_t\right), \quad t = 0, 1, 2, ... \\ & \tilde{z}_0 \in \tilde{\mathbb{Z}} \end{aligned} \tag{4}$$

where $\tilde{\mathbb{Z}} \subseteq \mathbb{R}_+^2$. The parameter β is defined as:

$$\beta = \frac{(1+\alpha)^{(1-\eta)}}{1+\delta} < 1 \tag{5}$$

where the inequality follows from the assumptions that $\eta > 0$ and $\delta \geq \alpha$. The redefined current-period utility, $\tilde{u}$, is given by:

$$\tilde{u}\left(\tilde{z}_t, \tilde{z}_{t+1}\right) = \begin{cases} \frac{(1+\alpha)^{(1-\eta)}}{1-\eta}\left[\frac{(1+r)\tilde{s}_t + y\tilde{\Theta}_t}{1+\alpha} - \tilde{s}_{t+1} - \tilde{\omega}(\tilde{\Theta}_{t+1}, \tilde{\Theta}_t)\right]^{1-\eta}, & \eta > 0 \wedge \eta \neq 1 \\ \log\left[\frac{(1+r)\tilde{s}_t + y\tilde{\Theta}_t}{1+\alpha} - \tilde{s}_{t+1} - \tilde{\omega}(\tilde{\Theta}_{t+1}, \tilde{\Theta}_t)\right], & \eta = 1 \end{cases} \tag{6}$$

where the expression in square brackets is $\tilde{c}_{t+1}$. The correspondence $\tilde{\Gamma}$ is defined as:

$$\tilde{\Gamma}\left(\tilde{z}_t\right) = \left\{ \tilde{z}_{t+1} \in \tilde{\mathbb{Z}} : \begin{array}{l} \tilde{\Theta}_{t+1} \in \left\{(1+\alpha)^{-1}\tilde{\Theta}_t, (1+\alpha)^{-1}\theta_0\right\} \\ \tilde{s}_{t+1} \leq \frac{(1+r)\tilde{s}_t + y\tilde{\Theta}_t}{1+\alpha} - \tilde{\omega}(\tilde{\Theta}_{t+1}, \tilde{\Theta}_t) \end{array} \right\} \tag{7}$$

Finally, the cost of investment, $\tilde{\omega}$, becomes:

$$\tilde{\omega}(\tilde{\Theta}_{t+1}, \tilde{\Theta}_t) = (1+\alpha)^{-1}\left[\omega_1\theta_0 + \omega_2(\theta_0 - \tilde{\Theta}_t)\right] \tag{8}$$

if $\tilde{\Theta}_{t+1} = (1+\alpha)^{-1}\theta_0$, and zero otherwise.

Now, consider the set $\tilde{\mathbb{Z}}$. In the most general case, $\tilde{\mathbb{Z}}$ is just $\mathbb{R}_+^2$. And since the normalised problem in (4)-(8) is well defined for all $\tilde{z}_0 \in \mathbb{R}_+^2$, a unique supremum function $\tilde{v}^* : \mathbb{R}_+^2 \to \mathbb{R}$ exists.[4]

However, a sequence problem of the above type does typically not allow for an analytical solution. Therefore, dynamic programming techniques and numerical solution methods must be applied if a solution to the problem is to be further characterised. For this purpose, $\tilde{\mathbb{Z}}$ must be restricted to some compact set in $\mathbb{R}_+^2$. The following Proposition shows that such a restriction is feasible without violating the original formulation of the problem.

Proposition 1 $\tilde{s}_t \leq \max\left\{\tilde{s}_{t-1}, (1+\alpha)^{-1}(\omega_1+\omega_2)\,\theta_0\right\}$ *for all* $t \geq 1$.

Proof. See Appendix. ■

Proposition 1 implies that given an initial level of (normalised) savings which exceeds the maximum (normalised) cost of investment, it will never be optimal to choose a higher level of (normalised) savings in the future. The proof of Proposition 1 relies on the assumption that $\delta \geq r > \alpha$, and it implies that for any choice of $\tilde{s}^{max} \geq (1+\alpha)^{-1}(\omega_1+\omega_2)\,\theta_0$, a set $\tilde{\mathbb{Z}}$ can be constructed as:

$$\tilde{\mathbb{Z}} = \left\{ (\tilde{s}, \tilde{\Theta}) \in \mathbb{R}_+^2 : \begin{array}{c} \tilde{\Theta} \in \left[0, (1+\alpha)^{-1}\theta_0\right] \\ \tilde{s} \in \left[0, \tilde{s}^{max}\right] \end{array} \right\} \tag{9}$$

such that for any $\tilde{z} \in \tilde{\mathbb{Z}}$, the range of the correspondence $\tilde{\Gamma}$ can be restricted to the compact set $\tilde{\mathbb{Z}}$ without loss of generality.

2.3 The Dynamic Programming Approach

To solve the problem by use of numerical methods, it is useful to work with a formulation of the problem in terms of dynamic programming. Thus, consider the following functional equation:

$$\tilde{v}(\tilde{z}) = \sup_{\tilde{z}' \in \tilde{\Gamma}(\tilde{z})} \left[\tilde{u}(\tilde{z}, \tilde{z}') + \beta \tilde{v}(\tilde{z}')\right] \quad all \quad \tilde{z} \in \tilde{\mathbb{Z}} \tag{10}$$

[4] This follows from the fact that: i) $\tilde{\Gamma}(\tilde{z}_t)$ is non-empty for all $\tilde{z}_t \in \mathbb{R}_+^2$; ii) $0 < \beta < 1$; and iii) current-period utility, $\tilde{u}$, is bounded from below by 0 when $\eta < 1$, and bounded from above by 0 when $\eta > 1$. For the case of $\eta = 1$, it is easy to show that: $\sum_{t=0}^{\infty} \beta^t \tilde{u}(\tilde{c}_{t+1}) \leq \sum_{t=0}^{\infty} \beta^t \tilde{u}(k^t \tilde{c}_1)$ for some positive constant k and $\tilde{c}_1 = (1+\alpha)^{-1}(1+r)\,\tilde{s}_0 + (1+\alpha)^{-2} y\theta_0$. This implies that $\sum_{t=0}^{\infty} \beta^t \tilde{u}(\tilde{c}_{t+1}) \leq (1-\beta)^{-1}\log(\tilde{c}_1) + \beta(1-\beta)^{-2}\log(k)$, which serves as an upper bound for the value function when $\eta = 1$.

where $\tilde{u}(\tilde{z}, \tilde{z}')$ and $\tilde{\Gamma}(\tilde{z})$ are given by (6) and (7), respectively, and $\tilde{\mathbb{Z}}$ is given by (9). Define the mapping $T : \tilde{V} \to \tilde{V}$ by:

$$(T\tilde{v})(\tilde{z}) = \sup_{\tilde{z}' \in \tilde{\Gamma}(\tilde{z})} [\tilde{u}(\tilde{z}, \tilde{z}') + \beta \tilde{v}(\tilde{z}')] \tag{11}$$

where $\tilde{V}$ is some space of functions. Thus, the mapping T associates with each function $\tilde{v} \in \tilde{V}$ the function $T\tilde{v} \in \tilde{V}$.

The idea underlying the solution method is to start with an initial guess on a value function, $\tilde{v}_0 : \tilde{\mathbb{Z}} \to \mathbb{R}$, and substitute it into the right-hand side of (11). The maximisation then yields a new value function, $\tilde{v}_1 = T\tilde{v}_0$, which again can be substituted into the right-hand side of (11). Under certain conditions, such successive iterations will eventually lead to at least an approximation of the true value function, $\tilde{v}^*$. The following Proposition states under which conditions this will actually be the case when $\eta < 1$.

Proposition 2 *Let $\eta < 1$. In addition, let $\tilde{\mathbb{Z}}$ be given by (9) with $\tilde{s}^{max} \geq (1+\alpha)^{-1}(\omega_1 + \omega_2)\theta_0$, and let $\tilde{u} : \tilde{\mathbb{Z}} \to \mathbb{R}$ and $\tilde{\Gamma} : \tilde{\mathbb{Z}} \to \mathbb{R}$ be given by (6) and (7), respectively. The mapping T defined in (11) is then a contraction mapping, taking the space of bounded functions $\tilde{v} : \tilde{\mathbb{Z}} \to \mathbb{R}$, with the sup norm, into itself: $T : B(\tilde{\mathbb{Z}}) \to B(\tilde{\mathbb{Z}})$. Furthermore, T has a unique fixed point, $\tilde{v}' \in B(\tilde{\mathbb{Z}})$, which is identical to the unique solution, $\tilde{v}^*$, of the sequence problem in (4)-(8). Finally, for all $\tilde{v}_0 \in B(\tilde{\mathbb{Z}})$:*

$$\|T^n \tilde{v}_0 - \tilde{v}'\| \leq \beta^n \|\tilde{v}_0 - \tilde{v}'\| \quad , \quad n = 0, 1, 2, ... \tag{12}$$

Proof. See Appendix. ■

Proposition 2 implies that starting with any initial guess on a value function in the space of bounded functions on $\tilde{\mathbb{Z}}$, successive iterations on the functional equation (11) will yield estimates which come closer and closer to the true value function, $\tilde{v}'$, which satisfies (10).

The fact that the feasible correspondence $\tilde{\Gamma}$ is not lower hemicontinuous at all points implies that standard results, such as those found in Stokey and Lucas (1989) where the mapping T takes the space of bounded, continuous functions into itself, cannot be applied. The value function will in general be discontinuous at some points when $\eta < 1$. Thus, Proposition 2 extends the standard contraction property to the case of merely bounded value functions.

To prove a similar property for the case of $\eta \geq 1$, it is necessary to deal with the problem that the value function will tend to minus infinity as savings, $\tilde{s}$, and technology, $\tilde{\Theta}$, tend to zero. This will violate the properties of the mapping, apart from being impossible to handle numerically. Technically, this can be dealt with by introducing a lower bound on either savings or technology. Thus, define $\tilde{\mathbb{Z}}_b$ by:

$$\tilde{\mathbb{Z}}_b = \left\{ (\tilde{s}, \tilde{\Theta}) \in \mathbb{R}_+^2 : \begin{array}{c} \tilde{\Theta} \in \left[\tilde{\Theta}^{min}, (1+\alpha)^{-1} \theta_0\right] \\ \tilde{s} \in \left[\tilde{s}^{min}, \tilde{s}^{max}\right] \end{array} \right\} \tag{13}$$

where $\tilde{s}^{min} \geq 0$ and $\tilde{\Theta}^{min} \geq 0$. If $\tilde{s}^{min} = \tilde{\Theta}^{min} = 0$, then $\tilde{\mathbb{Z}}_b$ and $\tilde{\mathbb{Z}}$ coincide.[5] Furthermore, the correspondence $\tilde{\Gamma}$ must be modified accordingly:

$$\tilde{\Gamma}_b(\tilde{z}_t) = \tag{14}$$
$$\left\{ \tilde{z}_{t+1} \in \tilde{\mathbb{Z}}_b : \begin{array}{l} \tilde{\Theta}_{t+1} = \left\{ \max\left\{ \tilde{\Theta}^{min}, (1+\alpha)^{-1} \tilde{\Theta}_t \right\}, (1+\alpha)^{-1} \theta_0 \right\} \\ \tilde{s}_{t+1} \leq \frac{(1+r)\tilde{s}_t + y\tilde{\Theta}_t}{1+\alpha} - \tilde{\omega}(\tilde{\Theta}_{t+1}, \tilde{\Theta}_t) \end{array} \right\}$$

Then it is straightforward to prove an analogue to Proposition 2 where the mapping T can be shown to operate on the space of bounded and *continuous* functions:

Proposition 3 *Let $\eta \geq 1$. In addition, let $\tilde{\mathbb{Z}}$ be given by $\tilde{\mathbb{Z}}_b$ in (13) with $\tilde{s}^{max} \geq (1+\alpha)^{-1} (\omega_1 + \omega_2) \theta_0$ and either: i) $\tilde{\Theta}^{min} > 0$; or ii) $\tilde{s}^{min} > 0$. Furthermore, let $\tilde{u} : \tilde{\mathbb{Z}} \to \mathbb{R}$ and $\tilde{\Gamma} : \tilde{\mathbb{Z}} \to \mathbb{R}$ be given by (6) and (14), respectively. The mapping T defined by (11) is then a contraction mapping, taking the space of bounded, continuous functions $\tilde{v} : \tilde{\mathbb{Z}} \to \mathbb{R}$, with the sup norm, into itself: $T : C(\tilde{\mathbb{Z}}) \to C(\tilde{\mathbb{Z}})$. Furthermore, T has a unique fixed point, $\tilde{v}' \in C(\tilde{\mathbb{Z}})$, which is identical to the unique solution, $\tilde{v}^*$, of the sequence problem in (4)-(8) with (7) replaced by (14). Finally, for all $\tilde{v}_0 \in C(\tilde{\mathbb{Z}})$:*

$$\|T^n \tilde{v}_0 - \tilde{v}'\| \leq \beta^n \|\tilde{v}_0 - \tilde{v}'\| \quad , \quad n = 0, 1, 2, ... \tag{15}$$

Proof. See Appendix. ■

Remark 4 $\exists\ \beta \in (0,1) \wedge \eta \in (0,1)$ *such that the value function is continuous and the contraction properties of Proposition 3 apply.*

The intuition underlying Proposition 3 and Remark 4 is that the cost of foregone consumption becomes relatively high compared to future utility when β is small and/or η is high. In this case, the agent will never choose to invest if this implies zero current consumption, and the proof of Proposition 3 can then be applied to show that the value function is continuous.

[5] When $\tilde{s}^{min} > 0$, Proposition 1 is modified such that $\tilde{s}_t \leq \max\{\tilde{s}_{t-1}, (1+\alpha)^{-1} (\omega_1 + \omega_2) \theta_0 + \tilde{s}^{min}\}$ for all $t \geq 1$.

What are the implications of i) and ii) in Proposition 3? First, imposing a lower bound on normalised technology implies that as the distance between installed and exogenous technology becomes large, where "large" depends on the value of $\tilde{\Theta}^{min}$ chosen, installed technology starts to grow at the same rate as exogenous technology. Secondly, assuming a lower bound on $\tilde{s}$ implies a lower (and growing) bound on s. The agent is required to hold a minimum amount of savings each period, and the size of this amount is given by $\tilde{s}^{min}$. That is, the agent is restricted from spending all his income on consumption and investments. Since $\tilde{s}^{min}$ can be chosen arbitrarily small, this restriction seems less at odds with the original formulation than assuming the existence of a lower bound on normalised technology. However, it turns out that the former restriction is the most natural to implement in the numerical simulations, as will become evident in Section 3.

What is the relationship between the solution to the sequence problem on $\tilde{\mathbb{Z}}$ and $\tilde{\mathbb{Z}}_b$?[6] Consider how the value function behaves as the lower bounds on savings and technology tend to zero, i.e. as $\tilde{\mathbb{Z}}_b$ is extended to $\tilde{\mathbb{Z}}$. Since the value function must be non-decreasing in both $\tilde{s}$ and $\tilde{\Theta}$, it will be smaller at the extra points included in the state space, and decreasing towards minus infinity as $\tilde{\mathbb{Z}}_b$ is increased to $\tilde{\mathbb{Z}}$. By assumption, it will take on the value minus infinity at $\tilde{s} = \tilde{\Theta} = 0$ since the only feasible strategy here is to consume zero for all future periods. However, as $\tilde{\Theta}^{min}$ is decreased, the value function will also change at interior points in the state space, namely at those points where future dynamics will take the agent to the lower boundary of $\tilde{\Theta}$, i.e. points where the agent is "caught" in a poverty trap. Increasing the lower bound on $\tilde{\Theta}$ will tend to increase the value function at these points, since the consequences of a decreasing sequence of $\tilde{\Theta}$ become less severe. As $\tilde{s}^{min}$ is decreased, it will tend to increase the value function at *all* interior points, since a smaller value of $\tilde{s}^{min}$ corresponds to enlarging the options available to the agent at all points, and this will certainly not make him worse off. Thus, imposing a lower bound on savings tends to affect the value function more broadly, whereas a lower bound on technology will only affect values at those states belonging to the poverty trap.

2.4 Characterisation of the Solution

The investment decision of the model can be interpreted as a repeated optimal-stopping problem, where "stopping" corresponds to "changing technology".

[6] Remember that the sequence problem is always well defined on both $\tilde{\mathbb{Z}}$ and $\tilde{\mathbb{Z}}_b$ for all $\eta > 0$, and that the solution to the sequence problem will always satisfy the corresponding functional equations.

The difference between this problem and standard optimal-stopping problems is that the stopping problem in this model is a repeated decision problem. The option to invest is not "killed" when an investment is undertaken, but is immediately replaced by a new option. Furthermore, the stopping problem is an integral part of a more general utility maximisation problem and can therefore not be solved separately. However, the fact that the problem resembles an optimal-stopping problem can (in some cases) still be used to characterise the solution a priori; a characterisation which in turn can be used to speed up the subsequent numerical simulations. For this purpose, it is instructive to define normalised "cash on hand" at the beginning of period $t+1$ by:

$$\tilde{x}_t = (1+r)\,\tilde{s}_t + y\tilde{\Theta}_t$$

The value function can then alternatively be expressed with $\tilde{x}$ and $\tilde{\Theta}$ as arguments: $\tilde{v}(\tilde{x}, \tilde{\Theta})$. Now, let $\tilde{v}^{stop}(\tilde{x}, \tilde{\Theta})$ denote the optimal value obtained from investing in new technology in the current period. Similarly, let $\tilde{v}^{cont}(\tilde{x}, \tilde{\Theta})$ denote the optimal value of continuing with the installed technology. Then for the case of $\omega_2 = 0$, the following result can be established:

Proposition 5 *Let $\omega_2 = 0$. Given some feasible $\tilde{x} \geq \omega_1\theta_0 + (1+\alpha)\,\tilde{s}^{min}$, either:*

1. *$\tilde{v}^{cont}(\tilde{x}, \tilde{\Theta}) > \tilde{v}^{stop}(\tilde{x}, \tilde{\Theta})$ for all $\tilde{\Theta} \in [\tilde{\Theta}^{min}, (1+\alpha)^{-1}\,\theta_0]$; or*
2. *$\tilde{v}^{cont}(\tilde{x}, \tilde{\Theta}) < \tilde{v}^{stop}(\tilde{x}, \tilde{\Theta})$ for all $\tilde{\Theta} \in [\tilde{\Theta}^{min}, (1+\alpha)^{-1}\,\theta_0]$; or*
3. *$\exists\ \tilde{\Theta}' \in [\tilde{\Theta}^{min}, (1+\alpha)^{-1}\,\theta_0]$ such that: i) $\tilde{v}^{cont}(\tilde{x}, \tilde{\Theta}') = \tilde{v}^{stop}(\tilde{x}, \tilde{\Theta}')$; ii) $\tilde{v}^{cont}(\tilde{x}, \tilde{\Theta}) > \tilde{v}^{stop}(\tilde{x}, \tilde{\Theta})$ for any $\tilde{\Theta} \in (\tilde{\Theta}', (1+\alpha)^{-1}\,\theta_0]$; and iii) $\tilde{v}^{cont}(\tilde{x}, \tilde{\Theta}) < \tilde{v}^{stop}(\tilde{x}, \tilde{\Theta})$ for any $\tilde{\Theta} \in [\tilde{\Theta}^{min}, \tilde{\Theta}')$.*

where $\tilde{\Theta}^{min} \geq 0$ and $\tilde{s}^{min} \geq 0$.

Proof. See Appendix. ■

Proposition 5 says that for those cash levels where both continuing with the installed technology and investing in new technology might be optimal, depending on the level of $\tilde{\Theta}$, there will be a single point, which will be referred to as the "stopping point", where the value of continuing exactly matches the value of investing. At higher levels of technology, continuing will be strictly preferred, whereas investing is strictly preferred at lower levels. The Proposition has several interesting implications.

First, given $\tilde{x}$, if the stopping point, $\tilde{\Theta}'$, can be localised, the optimal investment decisions for all remaining levels of installed technology follow directly.

Secondly, given $\tilde{x}$, the consumption choices will be independent of $\tilde{\Theta}$ for all $\tilde{\Theta} \leq \tilde{\Theta}'$. At these states, the agent will enter the following period with the highest possible technology, and since investment cost is independent of the level of installed technology, his current cash holdings net of investment cost will also be independent of $\tilde{\Theta}$. Thus, his optimal consumption choices in the current and the following periods must be identical for all $\tilde{\Theta} \leq \tilde{\Theta}'$.

Thirdly, it follows directly from the second implication that, given $\tilde{x}$, the value function must be constant for all $\tilde{\Theta} \leq \tilde{\Theta}'$. Together with the two first implications, this result can be used to speed up the numerical algorithm considerably, as described in Section 3.

Thus, the policy correspondence for investment will be single-valued on $\tilde{\mathbb{Z}}$, except possibly on a set of measure zero. For each value of $\tilde{x}$, there will be at most a single point where both investing and not investing might be optimal. However, since the numerical solutions will be restricted to a grid of technology levels (and cash levels), it is unlikely that they will come up with any such points. In case they do, it is more likely due to numerical imprecision than because an exact stopping point has been localised.

The policy correspondence for saving (consumption) is of course non-single valued at the same set of points, but it might in addition be non-single valued at a larger set. This is the case when there exist different, but equally optimal, future investment patterns. However, as for the investment correspondence, the numerical procedures are not likely to detect this.

The fact that policy functions do not generally exist precludes the use of analytical methods for solving the problem, and it restricts the range of feasible numerical techniques. Thus, more traditional value function iteration must be relied on. The approach used in Deaton (1991), where iterations are undertaken in terms of an indirect policy function, cannot be applied in the current set-up since policy functions do not generally exist. However, since the numerical solutions will in all likelihood yield single-valued policy correspondences, they will be referred to as policy *functions* in what follows.

Before turning to a description of the numerical techniques, consider the implications of the first order conditions of the problem. In each period, marginal utility of consumption must satisfy:

$$\tilde{u}'\left(\tilde{c}_t\right) = \max\left\{\tilde{u}'\left(\frac{\tilde{x}_{t-1}}{1+\alpha} - \tilde{s}^{min} - \tilde{\omega}(\tilde{\Theta}_t, \tilde{\Theta}_{t-1})\right), \beta\frac{1+r}{1+\alpha}\tilde{u}'\left(\tilde{c}_{t+1}\right)\right\} \quad (16)$$

That is, either no savings in excess of $\tilde{s}^{min}$ are carried forward, in which case consumption equals cash net of investment cost, or positive savings are carried forward, in which case the standard Euler property holds. Thus, since $\beta(1+r)(1+\alpha)^{-1} < 1$, the evolution of marginal utility of consumption (and thereby consumption) can be described by a smoothly increasing (decreasing) process which is reset whenever savings are brought to zero.

3 Numerical Simulations

Due to the contraction mapping properties of the optimisation problem, as stated in Proposition 2 and 3, it can be solved by use of numerical methods.[7] Choosing a relevant set of starting values for $\tilde{s}$, the set $\tilde{\mathbb{Z}}$ is given by (9) or (13). Supplying an initial, bounded function, $\tilde{v}_0$, into the functional equation (11), the maximisation will yield $\tilde{v}_1$ as an improved guess on $\tilde{v}'$ $(= \tilde{v}^*)$. Proceeding with such iterations, the estimate of $\tilde{v}'$ improves and converges towards $\tilde{v}'$. The iterative procedure is stopped when some convergence criterion is first satisfied. As a result, the value function $\tilde{v}(\tilde{z})$ and the corresponding policy functions: $\tilde{c}(\tilde{z})$, $\tilde{s}(\tilde{z})$, and $\tilde{\Theta}(\tilde{z})$ are obtained.

Several procedures for solving these kinds of problems exist. However, the non-smoothness and the potential discontinuity of the value function limit the range of appropriate methods. These issues will be discussed more fully in Section 5.4, but as a consequence of this complexity, a simple discrete grid value iteration procedure is applied in this paper.

3.1 Deterministic Algorithm

As mentioned in Section 2.4, the normalised value function can be expressed as a function of the current installed technology level, $\tilde{\Theta}$, and cash on hand, $\tilde{x}$. Now, a discrete grid over $\tilde{x}$ is created by letting $\tilde{x}^{max} = \tilde{s}^{max}(1+r) + y(1+\alpha)^{-1}\theta_0$ and choosing n_x equally spaced grid points.[8] For technology,

[7]The normalisation on θ_t is not necessary for obtaining a numerical approximation to the solution of the infinite horizon problem described above. One could truncate time and solve a finite time horizon problem involving three variables, θ_t, Θ_t, and s_t (or x_t). The number of periods, however, has to be fairly large (several hundreds), and because the transition functions are complex, the maximisation step will be slow. Hence an algorithm using this approach will not work fast on each time step. And the large number of periods needed for a good approximation is likely to be larger than the corresponding number of iterations in the infinite horizon approach described in the following.

[8]Note that choosing $\tilde{x}^{max} = \tilde{s}^{max}(1+r) + y(1+\alpha)^{-1}\theta_0$ yields grid points where the implicit value of $\tilde{s}$ exceeds $\tilde{s}^{max}$. In practice, these points are irrelevant since they will never be chosen from grid points satisfying $\tilde{s} \leq \tilde{s}^{max}$, and they can therefore be discarded ex-post. Alternatively, it is straightforward to show an analogue to Proposition 1 saying that $\tilde{x}_t \leq \max\left\{\tilde{x}_{t-1}, (1+\alpha)^{-1}\theta_0\left[(1+r)(\omega_1+\omega_2)+y\right]\right\}$ for all $t \geq 1$, and this directly

$\tilde{\Theta}$, it seems natural to choose the grid as $(1+\alpha)^{-1}\theta_0$, $(1+\alpha)^{-2}\theta_0$, etc., since only prior to the first investment can installed technology take on values outside this grid. Furthermore, since the number of grid points must be finite, it automatically implies a lower bound on $\tilde{\Theta}$, thereby satisfying assumption i) in Proposition 3. Thus, a discrete grid set, $\tilde{\mathbb{Z}}_g = \tilde{\mathbb{Z}}_{g\theta} \times \tilde{\mathbb{Z}}_{gx}$, with dimensions $n_\theta \times n_x$ can be defined as:

$$\tilde{\mathbb{Z}}_g = \left\{ (\tilde{x}, \tilde{\Theta}) \in \mathbb{R}_+^2 : \begin{array}{l} \tilde{x} \in \left\{ \frac{\tilde{x}^{max}}{n_x}, \frac{2\tilde{x}^{max}}{n_x}, ..., \tilde{x}^{max} \right\} = \tilde{\mathbb{Z}}_{gx} \\ \tilde{\Theta} \in \left\{ \frac{\theta_0}{(1+\alpha)^{n_\theta}}, \frac{\theta_0}{(1+\alpha)^{n_\theta - 1}}, ..., \frac{\theta_0}{(1+\alpha)^1} \right\} = \tilde{\mathbb{Z}}_{g\theta} \end{array} \right\}$$

At each grid point, $\tilde{z}_g = (\tilde{x}_g, \tilde{\Theta}_g)$, in every iteration, the algorithm solves the following optimisation problem:

$$\tilde{v}_{n+1}(\tilde{z}_g) = \max\left\{ \tilde{v}_{n+1}^{stop}(\tilde{z}_g), \tilde{v}_{n+1}^{cont}(\tilde{z}_g) \right\}$$

where $\tilde{v}_{n+1}^{cont}$ is the optimal value of continuing with the installed technology given by:

$$\tilde{v}_{n+1}^{cont}(\tilde{z}_g) = \max_{\tilde{x}_g' \in \tilde{\Gamma}_g^{cont}(\tilde{z}_g)} \left[\tilde{u}(\tilde{c}') + \beta \tilde{v}_n(\tilde{z}_g') \right] \quad all \quad \tilde{z}_g \in \tilde{\mathbb{Z}}_g \tag{17}$$

with $\tilde{\Theta}_g'$ given by:

$$\tilde{\Theta}_g' = \max\left\{ (1+\alpha)^{-1}\tilde{\Theta}_g, (1+\alpha)^{-n_\theta}\theta_0 \right\}$$

and $\tilde{c}'$:

$$\tilde{c}' = \frac{\tilde{x}_g}{(1+\alpha)} - \frac{\tilde{x}_g' - y\tilde{\Theta}_g'}{(1+r)}$$

whereas $\tilde{\Gamma}_g^{cont}$ is defined as:

$$\tilde{\Gamma}_g^{cont}(\tilde{z}_g) = \left\{ \tilde{x}_g' \in \tilde{\mathbb{Z}}_{gx} : y\tilde{\Theta}_g' \leq \tilde{x}_g' \leq \frac{(1+r)\tilde{x}_g}{1+\alpha} + y\tilde{\Theta}_g' \right\}$$

ensuring that $\tilde{c}' \geq 0$ and $\tilde{s}' \geq 0$.

Similarly, $\tilde{v}_{n+1}^{stop}$ is the optimal value of changing technology given by:

$$\tilde{v}_{n+1}^{stop}(\tilde{z}_g) = \max_{\tilde{x}_g' \in \tilde{\Gamma}_g^{stop}(\tilde{z}_g)} \left[\tilde{u}(\tilde{c}') + \beta \tilde{v}_n(\tilde{z}_g') \right] \tag{18}$$

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justifies the imposition of an upper bound on $\tilde{x}$.

for all $\tilde{z}_g$ with $\tilde{\Gamma}_g^{stop}(\tilde{z}_g) \neq \emptyset$. In this case, $\tilde{\Theta}'_g = (1+\alpha)^{-1}\theta_0$ and:

$$\tilde{c}' = \frac{\tilde{x}_g - \omega_1\theta_0 - \omega_2(\theta_0 - \tilde{\Theta}_g)}{1+\alpha} - \frac{\tilde{x}'_g - y\tilde{\Theta}'_g}{1+r}$$

whereas $\tilde{\Gamma}_g^{stop}$ is defined by:

$$\tilde{\Gamma}_g^{stop}(\tilde{z}_g) = \left\{\tilde{x}'_g \in \tilde{\mathbb{Z}}_{gx} : y\tilde{\Theta}'_g \leq \tilde{x}'_g \leq \frac{(1+r)\left(\tilde{x}_g - \omega_1\theta_0 - \omega_2(\theta_0 - \tilde{\Theta}_g)\right)}{1+\alpha} + y\tilde{\Theta}'_g\right\}$$

In the case of $\omega_2 = 0$, use of Proposition 5 can be made to speed up the maximisation step. In each iteration, the maximisation is performed first for the highest possible level of installed technology, $(1+\alpha)^{-1}\theta_0$, starting at the lowest possible level of cash on hand and proceeding upwards in cash to $\tilde{x}^{max}$. At each grid point, (17) and (18) are solved. The algorithm then proceeds to the next technology level, $(1+\alpha)^{-2}\theta_0$, and performs the same calculations – with one important exception: At this and lower technology levels in the algorithm, it is checked at each cash level larger than the investment cost whether it is optimal to invest at the preceding higher technology level with identical $\tilde{x}$. If this is the case, Proposition 5 implies that it must also be optimal at the current state, and that the optimal consumption levels must be identical. Therefore, the maximisation step is skipped for these points; the optimal choices for the higher technology level and same cash level are simply copied. This implies that for most of the states where investment is optimal, maximisation can be avoided. This reduces the computation time by 50-70%.

Through each iteration step, the value functions converge due to the contraction mapping properties of the problem. The iteration procedure is stopped when a measure of the change from $\tilde{v}_n(\tilde{z}_{ij})$ to $\tilde{v}_{n+1}(\tilde{z}_{ij})$ is below some pre-specified convergence criterion, ε. The measure chosen, Ψ, is the average, relative, squared deviation over all grid points:

$$\Psi = \frac{1}{n_\theta n_x}\sum_{i=1}^{n_\theta}\sum_{j=1}^{n_x}\frac{\left(\tilde{v}_{n+1}(\tilde{z}_{ij}) - \tilde{v}_n(\tilde{z}_{ij})\right)^2}{|\tilde{v}_n(\tilde{z}_{ij})|} \tag{19}$$

There are three motivations for this choice. First, to have a single criterion for the entire grid set. Secondly, because the solution may be discontinuous, the value function may at these points change a lot from iteration to iteration, in particular in the beginning. Since it is needed to determine the point of discontinuity quite precisely, more weight is put on avoiding large changes than minor ones. Hence the squared deviations. Finally, the measure should be independent of the exact level of the value function and therefore the squared deviations are scaled by $|\tilde{v}_n(\tilde{z}_{ij})|$.

With a value function estimate in the range of 0 to 35, setting $\varepsilon \leq 0.00001$ implies that the iterations are stopped when the average absolute change is less than 0.015.

The grid approximation implies that consumption is in each state restricted to a limited set of values. The coarseness of this set depends on the coarseness of the grid in cash on hand. Thus, consumption cannot be optimised freely and hence the value function estimate will inevitably be an underestimate of the true value function. Furthermore, due to the discreteness of the feasible consumption set, non-optimal, non-smooth evolution in consumption policy is likely to appear in the numerical results in the cases where optimal consumption levels have a magnitude similar to the minimum distance between possible consumption levels.

Given the obtained value and policy functions, time series simulations can be produced by picking initial states of interest.

The program that solves the optimisation problem and produces time series simulations of optimal consumption, technology, and saving levels is written using Borland's Turbo Pascal. The program is run as a control application in Borland's Delphi 3. Depending on parameters and grid coarseness, the results of the deterministic problem can be obtained in a few minutes on a PC of type Pentium III with a 700-MHz processor and 256 Mb RAM.

3.2 Simulation Results

The parameters in the deterministic model are: η, α, r, δ, ω_1, ω_2, y, and θ_0. However, since it is virtually impossible to present results for more than a small number of the derived parameter combinations, a non-negligible amount of discretion is unavoidable in the process of selecting the simulations to run and the results to present. The task is to pick out a few interesting and/or representative values of the parameters. Furthermore, decisions must be made regarding the mode of presentation such that the mechanisms and the results of the model are most clearly revealed.

TABLE I

BENCHMARK PARAMETER VALUES IN DETERMINISTIC MODEL

η	α	r	δ	ω_1	ω_2	y	θ_0
0.5	0.03	0.05	0.07	2	0	1	1

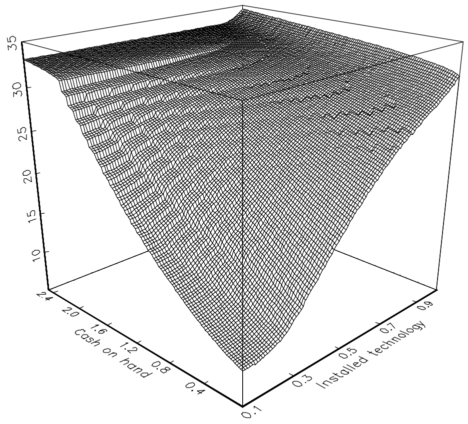

Figure 2: Value function

The benchmark values of the parameters to be used in the simulations of the deterministic model are listed in Table I. Deviations from these values will be explicitly specified when considered. Furthermore, figures will be presented in terms of normalised variables, unless otherwise stated.

To illustrate some of the practical problems involved in the numerical iterations on the value function, Figure 2 contains the resulting value function from an iteration using the benchmark parameter values. Figure 3 presents the same value function for a smaller range of cash values.

It should be apparent from these Figures that the value function does not display any of the "standard" properties of continuity, concavity, and differentiability. The largest discontinuities are found for small levels of installed technology along the line where cash on hand is just sufficient to cover the cost of an investment, i.e. at $\tilde{x} = 2$. As $\tilde{x}$ reaches 2 from below, the agent receives an extra option, namely the option to invest in the current period.[9] For small levels of installed technology, using this option strictly dominates any other feasible strategy, even though it requires zero consumption in the current period. Therefore, the value function "jumps" at these points. At higher levels of installed technology, the agent prefers to postpone the investment one or more periods and to use his current technology to enjoy reasonable levels of consumption until then.

The discontinuities at $\tilde{x} = 2$ "spill back" and yield other discontinuities at lower cash levels. More precisely, discontinuities can be located at combinations of $\tilde{x}$ and $\tilde{\Theta}$ where foregoing consumption for two or more periods will precisely enable the agent to undertake an investment at the end of these

[9] More formally, the feasible set is not lower hemicontinuous at $\tilde{x} = 2$.

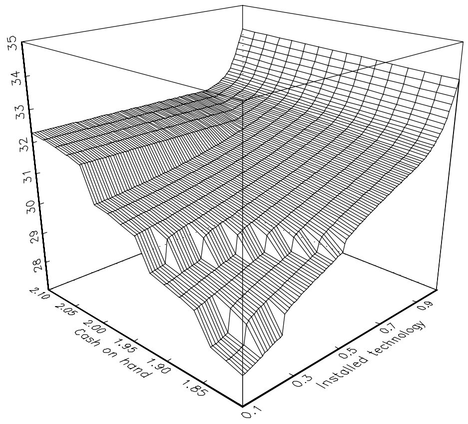

Figure 3: Value function

periods. Eventually, these discontinuities will die out as the cost of foregone consumption starts to exceed the gains from a (less distant) future investment. The resulting shape of the value function resembles that of a "winding staircase".

Thus, the discontinuities are caused by the assumptions of indivisible investments and credit constraints, together with the discrete nature of time in the model. Their magnitudes depend on the parameters determining the cost of investment, i.e. ω_1 and ω_2, and the relative importance of current and future consumption, i.e. η and δ. As stated in Proposition 3 and Remark 4, there exist values of the parameters at which the discontinuities disappear. Though continuous in these cases, the value function will still be highly non-smooth, implying that iteration methods relying on parametric approximations of the value function are virtually impossible to apply.

From an economic point of view, the policy functions for consumption, saving, and investment are more interesting. As for the value function, these could be visualised in three dimensions. However, since they turn out to be even less smooth than the value function, it is more instructive to use contour plots.

Figure 4 presents consumption functions for four different levels of installed technology. The imprecision arising from the grid approximation is reflected in the small upward steps of the functions. The consumption function for $\tilde{\Theta} = 0.4$ clearly illustrates the discontinuity of the value function at $\tilde{x} = 2$, where consumption suddenly drops to zero.

While the discontinuities of the value function disappeared at higher levels of technology and cash on hand, this is not the case for the policy function.

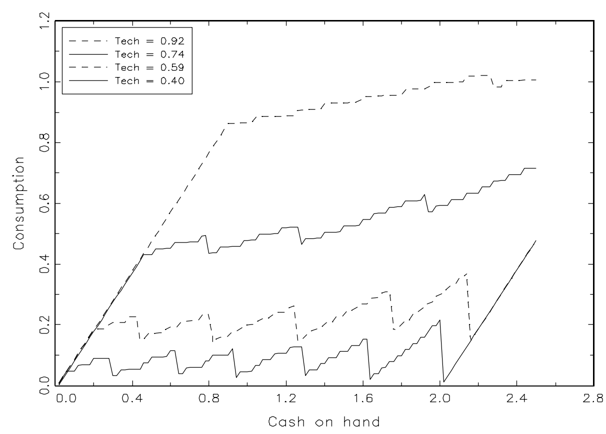

Figure 4: Consumption

Instead, a highly "sawtoothed" pattern emerges. Consider, e.g., the case of $\tilde{\Theta} = 0.59$. In this case, there is a discontinuity at $\tilde{x} > 2$. The intuition is that the relative cost of zero consumption is too high at $\tilde{\Theta} = 0.59$. Thus, the agent prefers to postpone the investment at $\tilde{x} = 2$. However, as $\tilde{x}$ is further increased, it will eventually reach a level where the value of stopping equals the value of continuation. This is the optimal-stopping nature of the problem. But though investing this period becomes optimal, it still implies lower current consumption in order to finance the investment. The remaining discontinuities in Figure 4 can be explained in a similar way: As cash increases, it becomes optimal to decrease the remaining time to an investment, even though this implies lower consumption in the remaining periods before the investment.

Up to a certain level of cash, $\tilde{x}^*$, everything is consumed, $\tilde{c} = (1+\alpha)^{-1}\tilde{x}$. This level depends on installed technology and is always lower than the level of income next period. It thus reflects the consumption smoothing motive of the agent.

The case of $\tilde{\Theta} = 0.68$ is considered more fully in Figure 5. It follows from the Figure that investing becomes optimal around $\tilde{x} = 2.4$, though this implies lower current consumption. As cash is further increased to a level around 3.8, the investment ceases to be optimal. At yet higher levels of cash, investing is once more the optimal strategy. Explaining these changes in strategy is not straightforward. They reflect the relative effectiveness of savings and investments as means of creating and redistributing income over time, taking into account that the investment decision today is only part of a larger investment plan.

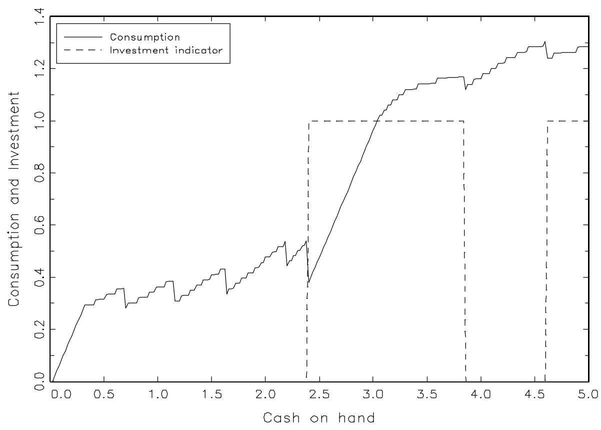

Figure 5: Consumption and investment, $\tilde{\Theta} = 0.68$

It should be stressed that Figure 5 presents a rather special case; a level of installed technology where investing in new technology is just about to become worthwhile. For most values of installed technology, there will be a single optimal stopping point.

Clearly, if the initial levels of cash and installed technology are too low, the agent might not find it worthwhile to save for a future investment. The cost in terms of foregone consumption is too large, and the agent instead maximises utility with respect to consumption given a future constant stream of non-normalised income. Due to impatience, the optimal strategy will then be to run down any initial assets, and then equalise consumption to income in all future periods.[10] Thus, the poverty trap is in some sense "voluntary"; the agent "chooses" not to save for future investments.

The poverty trap is therefore of the same nature as the one in Galor and Zeira (1993), where poor individuals cannot overcome the threshold needed for investment, but different from the type in Barro and Sala-I-Martin (1995), which is due to scale properties of the production function.

Figure 6 presents the case of $\eta = 1.5$. The picture is virtually the same as before. However, two aspects deserve comments. First, for the lowest technology level, the presence of a poverty trap is visible. At a cash level around 1.1, consumption drops dramatically, reflecting that an agent in this state finds it worthwhile to escape the poverty trap by saving for a future investment. Secondly, consumption functions, though still sawtoothed, ap-

[10]In the numerical simulations, however, as normalised technology reaches the lower bound, non-normalised income starts to grow at the rate of α, implying a similar growth rate of non-normalised consumption.

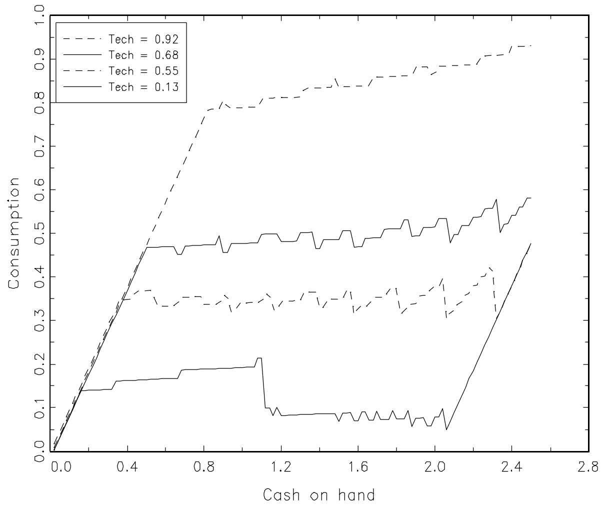

Figure 6: Consumption, $\eta = 1.5$

pear "flatter" than in the case of $\eta = 0.5$, reflecting the lower intertemporal elasticity of substitution, $1/\eta$. Thus, a smoother evolution of consumption is observed at $\eta = 1.5$, even though a higher η also works to decrease β.

Figure 7 presents simulations of consumption, savings, cash on hand, and installed technology. It appears that both consumption and savings follow a very cyclical pattern where high consumption is enjoyed at the beginning of a cycle, whereas savings accelerate towards the end of a cycle. These cyclical shapes reflect the countervailing forces of the accumulation motive and impatience. These motives interact to produce a cyclical pattern where most of the saving for an investment is undertaken in the periods immediately before the investment.

It is straightforward to check that an agent starting with $\tilde{s} = 0$ and $\tilde{\Theta} = (1+\alpha)^{-1}\theta_0$ will invest at a future point in time. This implies that any agent that finds it optimal to invest once will repeat investing at future points in time. Eventually, such "thrifty" agents will enter a deterministic investment cycle. To see this, remember that as long as the agent carries positive savings, (16) implies that $\tilde{c}$ (and c) must be declining over time. Thus, in order to increase consumption as technology increases, the agent must run down assets when an investment is undertaken, thereby resetting the Euler property of consumption paths. This implies that the normalised optimisation problem becomes identical after each new investment. Thus, a stable (or absorbing) cycle emerges in the simulations; a cycle which can be considered as the steady state dynamics of the model. Of course, the

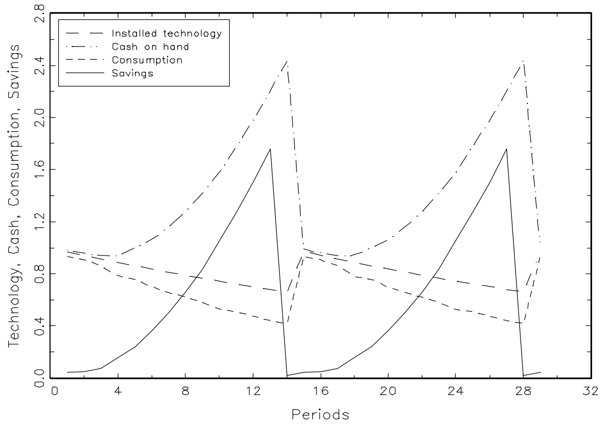

Figure 7: Simulations of technology, cash, consumption, and savings

properties of these cycles will depend crucially on the involved parameters. Table II presents characteristics of the cycles for varying values of η:

TABLE II

SENSITIVITY OF PERIOD LENGTH AND CONSUMPTION WITH RESPECT TO η

η	0.5	0.95	1.05	1.5
$\bar{p}$	14	14	15	15
$\bar{c}$	0.80	0.81	0.82	0.82

Note: Remaining parameter values as in Table I. $\bar{p}$ and $\bar{c}$ as defined in text.

where $\bar{p}$ is the investment period length, and $\bar{c}$ is average consumption out of income, defined as:

$$\bar{c} = \frac{1}{n}\sum\frac{\tilde{c}}{\tilde{y}} = \frac{1}{n}\sum\frac{c}{y} \tag{20}$$

where the summation is over the number of simulated periods, n.[11] It is seen that the length of the investment cycle increases as η increases. This is due to the lower intertemporal elasticity of substitution. The cost of foregone consumption in the current cycle is too high, causing the agent to choose a longer period between investments.

Figure 8 compares savings and consumption for $\eta = 0.5$ and $\eta = 1.5$. As mentioned above, the consumption profile is obviously smoother with higher

[11] In the deterministic model, it is in principle sufficient to sum over a single steady state cycle.

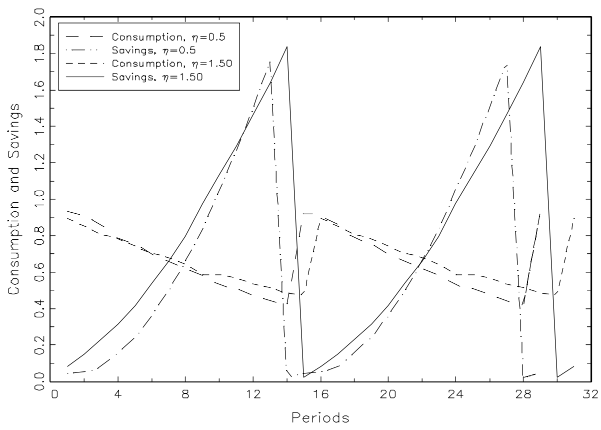

Figure 8: Simulations of consumption and savings, $\eta = 0.5$ and $\eta = 1.5$

η, i.e. savings are initially higher and do not accelerate as much as with $\eta = 0.5$.

Varying the parameter values in the cost function does not produce any surprising results, cf. Table III. Lowering the fixed cost of investment, ω_1, drastically reduces the time period between investments. However, adding a positive variable cost, ω_2, has only a minor effect on the average length.

TABLE III
AVERAGE LENGTH OF CYCLE

(ω_1, ω_2)	$(1,1)$	$(2,0)$	$(2,1)$
$\bar{p}$	8.94	13.97	14.75

Note: $\bar{p}$ as defined in text.

4 The Stochastic Model

This section presents a stochastic version of the model from the previous sections. Uncertainty is introduced through the income function given by:

$$y_t = m_t \Theta_t$$

or using normalised variables:

$$\tilde{y}_t = m_t \tilde{\Theta}_t$$

where m_t is an i.i.d. continuous, stochastic variable with compact support $\mathbb{M} = [\underline{m}, \overline{m}]$, where $\underline{m} \geq 0$, $\overline{m} < \infty$, $E(m) = \mu$, and $V(m) = \sigma^2$. The farmer is assumed to maximise expected utility.

Uncertainty could be introduced in a number of other ways, e.g. through the growth process of technology and/or the return to savings. However, the present type of uncertainty is sufficiently simple to allow most of the analytical results from the previous sections to go through without major changes. And still, the uncertainty introduces the precautionary saving motive.

The functional equation under uncertainty becomes:

$$\tilde{v}(\tilde{z}, m) = \max_{\tilde{z}' \in \tilde{\Gamma}(\tilde{z}, m)} \left[\tilde{u}(\tilde{z}, \tilde{z}', m) + \beta \int_{\underline{m}}^{\overline{m}} \tilde{v}(\tilde{z}', m')\, dF(m') \right] \tag{21}$$

where $\tilde{z} = (\tilde{s}, \tilde{\Theta})$, and F is the distribution function of m. Furthermore, $\tilde{u}$ is defined by:

$$\tilde{u}(\tilde{z}, \tilde{z}', m) = \begin{cases} \frac{(1+\alpha)^{(1-\eta)}}{1-\eta} \left[\frac{(1+r)\tilde{s} + m\tilde{\Theta}}{1+\alpha} - \tilde{s}' - \tilde{\omega}(\tilde{\Theta}', \tilde{\Theta}) \right]^{1-\eta}, & \eta > 0 \wedge \eta \neq 1 \\ \log \left[\frac{(1+r)\tilde{s} + m\tilde{\Theta}}{1+\alpha} - \tilde{s}' - \tilde{\omega}(\tilde{\Theta}', \tilde{\Theta}) \right], & \eta = 1 \end{cases} \tag{22}$$

where the expression in square brackets is $\tilde{c}_{t+1}$. β and $\tilde{\omega}(\tilde{\Theta}', \tilde{\Theta})$ are defined as in the deterministic case. The correspondence $\tilde{\Gamma} : \tilde{\mathbb{Z}} \times \mathbb{M} \to \tilde{\mathbb{Z}}$ is now defined by:

$$\tilde{\Gamma}(\tilde{z}, m) = \left\{ \tilde{z}' \in \tilde{\mathbb{Z}} : \begin{array}{l} \tilde{\Theta}' \in \left\{ (1+\alpha)^{-1} \tilde{\Theta}, (1+\alpha)^{-1} \theta_0 \right\} \\ \tilde{s}' \leq (1+\alpha)^{-1} \left[(1+r)\tilde{s} + m\tilde{\Theta} \right] - \tilde{\omega}(\tilde{\Theta}', \tilde{\Theta}) \end{array} \right\} \tag{23}$$

where $\tilde{\mathbb{Z}} \subseteq \mathbb{R}^2_+$.[12] As in the deterministic case, it is possible to restrict attention to a compact set:

Proposition 6 *$\exists\ \tilde{s}^{high} < \infty$ such that if $\tilde{s}_0 \leq \tilde{s}^{max}$ where $\tilde{s}^{max} \geq \tilde{s}^{high}$, then $\tilde{s}_t \leq \tilde{s}^{max}$ for all $t > 0$.*

Proof. See Appendix. ■

Proposition 6 says that when (normalised) savings are sufficiently large, the agent will choose to decrease savings next period, independent of the

[12] A sequence problem analogous to the one in the deterministic case can be formulated, but is omitted. Again, the value function $\tilde{v}^*$ will be well defined for all $(\tilde{z}_0, m_0) \in \mathbb{R}^2_+ \times \mathbb{M}$, since $\tilde{u}$ and $\tilde{\Gamma}$ satisfy the needed measurability requirements.

current income realisation, i.e. he will choose consumption and investment expenditures in excess of income and interest earned. The reason is that when savings are sufficiently large, they become the single determinant of consumption. Future investment cost and income realisations have negligible influence on future consumption, and hence uncertainty is of little importance. Due to relative impatience, it will always be optimal to decrease the level of savings in such a situation.

More formally, Proposition 6 implies that for any choice of $\tilde{s}^{max}$ above $\tilde{s}^{high}$, a compact set $\tilde{\mathbb{Z}}$ can be defined by (9) such that for any $\tilde{z} \in \tilde{\mathbb{Z}}$, the range of the correspondence Γ can be restricted to $\tilde{\mathbb{Z}}$ without loss of generality.

Define the mapping $T : \tilde{V} \to \tilde{V}$ by:

$$(T\tilde{v})(\tilde{z}, m) = \max_{\tilde{z}' \in \tilde{\Gamma}(\tilde{z},m)} \left[\tilde{u}(\tilde{z}, \tilde{z}', m) + \beta \int_{\underline{m}}^{\overline{m}} \tilde{v}(\tilde{z}', m')\, dF(m') \right] \qquad (24)$$

where $\tilde{V}$ is some space of functions. Then the following result can be established:

Proposition 7 *Let $\eta < 1$, and let $\tilde{\mathbb{X}} = \tilde{\mathbb{Z}} \times \mathbb{M}$ where $\tilde{\mathbb{Z}}$ is given by (9) with $\tilde{s}^{max}$ satisfying Proposition 6. In addition, let $\tilde{u} : \tilde{\mathbb{Z}} \times \tilde{\mathbb{Z}} \times \mathbb{M} \to \mathbb{R}$ and $\tilde{\Gamma} : \tilde{\mathbb{Z}} \times \mathbb{M} \to \tilde{\mathbb{Z}}$ be given by (22) and (23), respectively. The mapping T defined by (24) is then a contraction mapping, taking the space of bounded functions $\tilde{v} : \tilde{\mathbb{X}} \to \mathbb{R}$, with the sup norm, into itself: $T : B(\tilde{\mathbb{X}}) \to B(\tilde{\mathbb{X}})$. Furthermore, T has a unique fixed point, $\tilde{v} \in B(\tilde{\mathbb{X}})$, which corresponds to the unique solution of the corresponding sequence problem. Finally, for all $\tilde{v}_0 \in B(\tilde{\mathbb{X}})$:*

$$\|T^n \tilde{v}_0 - \tilde{v}\| \leq \beta^n \|\tilde{v}_0 - \tilde{v}\| \quad , \quad n = 0, 1, 2, ... \qquad (25)$$

Proof. See Appendix. ■

When $\eta \geq 1$, the state space must be redefined as in the deterministic case. Thus, $\tilde{\mathbb{Z}}$ must now be given by $\tilde{\mathbb{Z}}_b$ in (13). Furthermore, the feasible correspondence must be restricted to operate on this set:

$$\tilde{\Gamma}_b(\tilde{z}, m) = \qquad (26)$$

$$\left\{ \tilde{z}' \in \tilde{\mathbb{Z}}_b : \begin{array}{l} \tilde{\Theta}' \in \left\{ \max\left\{ \tilde{\Theta}^{min}, (1+\alpha)^{-1} \tilde{\Theta} \right\}, (1+\alpha)^{-1} \theta_0 \right\} \\ \tilde{s}' \leq \frac{(1+r)\tilde{s} + m\tilde{\Theta}}{1+\alpha} - \tilde{\omega}(\tilde{\Theta}', \tilde{\Theta}) \end{array} \right\}$$

Then the following result can be established:

Proposition 8 *Let* $\eta \geq 1$*, and let* $\tilde{\mathbb{X}} = \tilde{\mathbb{Z}} \times \mathbb{M}$ *where* $\tilde{\mathbb{Z}}$ *is given by* $\tilde{\mathbb{Z}}_b$ *in (13) with* $\tilde{s}^{max}$ *satisfying Proposition 6 and either: i)* $\tilde{\Theta}^{min} > 0$ *and* $\underline{m} > 0$*; or ii)* $\tilde{s}^{min} > 0$*. In addition, let* $\tilde{u} : \tilde{\mathbb{Z}} \times \tilde{\mathbb{Z}} \times \mathbb{M} \to \mathbb{R}$ *and* $\tilde{\Gamma} : \tilde{\mathbb{Z}} \times \mathbb{M} \to \tilde{\mathbb{Z}}$ *be given by (22) and (26), respectively. The mapping* T *defined by (24) is then a contraction mapping, taking the space of bounded, continuous functions* $\tilde{v} : \tilde{\mathbb{X}} \to \mathbb{R}$*, with the sup norm, into itself:* $T : C(\tilde{\mathbb{X}}) \to C(\tilde{\mathbb{X}})$*. Furthermore,* T *has a unique fixed point,* $\tilde{v} \in C(\tilde{\mathbb{X}})$*, which corresponds to the unique solution of the corresponding sequence problem. Finally, for all* $\tilde{v}_0 \in C(\tilde{\mathbb{X}})$*:*

$$\|T^n \tilde{v}_0 - \tilde{v}\| \leq \beta^n \|\tilde{v}_0 - \tilde{v}\| \quad , \quad n = 0, 1, 2, ... \tag{27}$$

Proof. See Appendix. ■

4.1 Characterisation of the Solution

As in the deterministic case, the value function can be expressed solely as a function of installed technology, $\tilde{\Theta}$, and cash on hand, $\tilde{x}$. And in the case of $\omega_2 = 0$, a result similar to the one in Proposition 5 holds:

Proposition 9 *Let* $\omega_2 = 0$*. Given some feasible* $\tilde{x} \geq \omega_1 \theta_0 + (1 + \alpha)\, \tilde{s}^{min}$*, either:*

1. $\tilde{v}^{cont}(\tilde{x}, \tilde{\Theta}) > \tilde{v}^{stop}(\tilde{x}, \tilde{\Theta})$ *for all* $\tilde{\Theta} \in [\tilde{\Theta}^{min}, (1 + \alpha)^{-1} \theta_0]$*; or*
2. $\tilde{v}^{cont}(\tilde{x}, \tilde{\Theta}) < \tilde{v}^{stop}(\tilde{x}, \tilde{\Theta})$ *for all* $\tilde{\Theta} \in [\tilde{\Theta}^{min}, (1 + \alpha)^{-1} \theta_0]$*; or*
3. $\exists\ \tilde{\Theta}' \in [\tilde{\Theta}^{min}, (1 + \alpha)^{-1} \theta_0]$ *such that: i)* $\tilde{v}^{cont}(\tilde{x}, \tilde{\Theta}') = \tilde{v}^{stop}(\tilde{x}, \tilde{\Theta}')$*; ii)* $\tilde{v}^{cont}(\tilde{x}, \tilde{\Theta}) > \tilde{v}^{stop}(\tilde{x}, \tilde{\Theta})$ *for any* $\tilde{\Theta} \in (\tilde{\Theta}', (1 + \alpha)^{-1} \theta_0]$*; and iii)* $\tilde{v}^{cont}(\tilde{x}, \tilde{\Theta}) < \tilde{v}^{stop}(\tilde{x}, \tilde{\Theta})$ *for any* $\tilde{\Theta} \in [\tilde{\Theta}^{min}, \tilde{\Theta}')$*.*

where $\tilde{\Theta}^{min} \geq 0$ *and* $\tilde{s}^{min} \geq 0$*.*

Proof. See Appendix. ■

The implications for policy functions and value functions are similar to those of the deterministic case and can again be used to speed up the numerical solutions.

The evolution of marginal utility of consumption can in the stochastic case be described by:

$$\tilde{u}'(\tilde{c}_t) = \max \left\{ \tilde{u}' \left(\frac{\tilde{x}_{t-1}}{1 + \alpha} - \tilde{s}^{min} - \tilde{\omega}(\tilde{\Theta}_t, \tilde{\Theta}_{t-1}) \right), \beta \frac{1 + r}{1 + \alpha} E_t \tilde{u}'(\tilde{c}_{t+1}) \right\}$$

Again, either no savings in excess of $\tilde{s}^{min}$ are carried forward, in which case consumption equals cash net of investment cost, or positive savings are carried forward, in which case the standard Euler or martingale property holds. Thus, the evolution of marginal utility of consumption, and thereby consumption, can be described by a martingale process which loses its memory whenever savings are brought to zero.

5 Numerical Simulations

Proposition 7 and 8 imply that the numerical iterations can be undertaken on the compact set $\tilde{\mathbb{Z}} \times \mathbb{M}$ with $\tilde{\mathbb{Z}}$ given by (9) or (13) for some $\tilde{s}^{max}$ satisfying Proposition 6. However, $\tilde{s}^{max}$ might have to be very large to satisfy Proposition 6, a problem that will be addressed in Section 5.2 below. First, a more detailed description of the stochastic algorithm will be presented.

5.1 Stochastic Algorithm

Let $\tilde{\mathbb{Z}}_g = \tilde{\mathbb{Z}}_{gx} \times \tilde{\mathbb{Z}}_{g\theta}$ be a $n_x \times n_\theta$ grid defined as in the deterministic case. That is, for a given value of $\tilde{x}^{max}$:[13]

$$\tilde{\mathbb{Z}}_g = \left\{ (\tilde{x}, \tilde{\Theta}) \in \mathbb{R}^2_+ : \begin{array}{l} \tilde{x} \in \left\{ \frac{\tilde{x}^{max}}{n_x}, \frac{2\tilde{x}^{max}}{n_x}, ..., \tilde{x}^{max} \right\} = \tilde{\mathbb{Z}}_{gx} \\ \tilde{\Theta} \in \left\{ \frac{\theta_0}{(1+\alpha)^{n_\theta}}, \frac{\theta_0}{(1+\alpha)^{n_\theta - 1}}, ..., \frac{\theta_0}{(1+\alpha)^1} \right\} = \tilde{\mathbb{Z}}_{g\theta} \end{array} \right\}$$

Stochastic income is introduced into the model through the variable m_t, which is given a truncated normal distribution with mean equal to one, standard deviation $\sigma \in [0, 0.3]$, and truncated at three standard deviations from the mean. The restrictions on σ ensure that $\underline{m} > 0$.

Given the distribution of m_t, normalised income, $m_t\tilde{\Theta}_t$, is approximated for each value of $\tilde{\Theta}$ in the grid $\tilde{\mathbb{Z}}_{g\theta}$. This is done by constructing n_θ vectors: $D(\tilde{y} \mid \tilde{\Theta}, \sigma, g_x)$ for $\tilde{\Theta} \in \tilde{\mathbb{Z}}_{g\theta}$. Each vector gives the concentrated probabilities of a grid of income realisations. The concentrated probability is obtained by integrating over the density area on both sides of the grid point. The distance between the income realisations in the grid corresponds to the distance between grid points in the grid over cash, $\tilde{\mathbb{Z}}_{gx}$. And since income realisations have been restricted to an interval of width $2 \cdot 3\sigma\tilde{\Theta}$, the number of elements

[13] As in the deterministic case, it is straightforward to prove that an upper bound on $\tilde{x}$ exists.

in each vector, $D(\tilde{y} \mid \tilde{\Theta}, \sigma, g_x)$, is given by:

$$n_y = 2 \cdot \text{integer}\left(\frac{3\sigma\tilde{\Theta}}{g_x}\right) + 1 \tag{28}$$

where $g_x = \tilde{x}^{max}/n_x$ is the distance between grid points in the cash grid, $\tilde{\mathbb{Z}}_{gx}$.

At each grid point, $\tilde{z}_g \in \tilde{\mathbb{Z}}_g$, in every iteration, the stochastic algorithm solves:

$$\tilde{v}_{n+1}(\tilde{z}_g) = \max\left\{\tilde{v}^{stop}_{n+1}(\tilde{z}_g), \tilde{v}^{cont}_{n+1}(\tilde{z}_g)\right\}$$

where $\tilde{v}^{cont}_{n+1}$ is the optimal (expected) value of continuing with the installed technology, and $\tilde{v}^{stop}_{n+1}$ is the optimal (expected) value of changing technology.

$\tilde{v}^{cont}_{n+1}$ is computed in the following way: Installed technology next period is given by: $\tilde{\Theta}' = \max\{(1+\alpha)^{-1}\tilde{\Theta}, (1+\alpha)^{-n_\theta}\theta_0, \}$. Optimisation can then be undertaken by finding the optimal level of *expected* cash on hand next period, $\tilde{x}'_e$, since the *distribution* of cash on hand next period follows from $\tilde{x}'_e$ and the distribution of income, $D(\tilde{y} \mid \tilde{\Theta}, \sigma, g_x)$.

More formally, given $\tilde{\Theta}'$, let $\tilde{x}'_{low}$ be the lowest possible cash state next period defined by:

$$\tilde{x}'_{low} = \text{integer}\left(\frac{\tilde{x}'_e - 3\sigma\tilde{\Theta}'}{g_x} + 1\right) \cdot g_x$$

and let the feasible set of choices for expected cash on hand next period be given by:

$$\tilde{\Phi}^{cont}_g(\tilde{z}_g) = \left\{\tilde{x}'_g \in \tilde{\mathbb{Z}}_{gx} : \tilde{\Theta}' \leq \tilde{x}'_g \leq \frac{\tilde{x}_g(1+r)}{1+\alpha} + \tilde{\Theta}'\right\}$$

when continuation has been chosen. Then for all $\tilde{z}_g \in \tilde{\mathbb{Z}}_g$, $\tilde{v}^{cont}_{n+1}$ is given by:

$$\tilde{v}^{cont}_{n+1}(\tilde{z}_g) = \max_{\tilde{x}'_e \in \tilde{\Phi}^{cont}_g(\tilde{z}_g)} \left\{\tilde{u}(\tilde{c}') + \beta \sum_{j=0}^{n_y-1} \tilde{v}_n\left(\tilde{\Theta}', \min\{\tilde{x}'_{low} + j \cdot g_x, \tilde{x}^{max}\}\right) \times p\left(\tilde{x}'_{low} + j \cdot g_x \,\middle|\, \tilde{z}_g, \tilde{\Theta}', \tilde{x}'_e\right)\right\} \tag{29}$$

where the probability $p(\tilde{x}'_{low} + j \cdot g_x \mid \tilde{z}_g, \tilde{\Theta}', \tilde{x}'_e)$ over $\tilde{x}'$ is implied by the choice of expected cash, $\tilde{x}'_e$, and is given by the $(j+1)^{th}$ element of the corresponding probability vector of income:

$$p\left(\tilde{x}'_{low} + j \cdot g_x \,\middle|\, \tilde{z}_g, \tilde{\Theta}', \tilde{x}'_e\right) = D_{j+1}\left(\tilde{y} \,\middle|\, \tilde{\Theta}', \sigma, g_x\right)$$

The "min"-operator in (29) is included to ensure that the value function is not evaluated outside the grid. In case the choice of $\tilde{x}'_e$ implies that positive probabilities are assigned to values of $\tilde{x}'$ above $\tilde{x}^{max}$, the probability is concentrated on $\tilde{x}^{max}$ instead. Finally, savings and consumption can be derived as residuals: $\tilde{s}' = (1+r)^{-1}(\tilde{x}'_e - \tilde{\Theta}')$ and $\tilde{c}' = (1+\alpha)^{-1}\tilde{x}_g - \tilde{s}'$.

The value of changing technology, $\tilde{v}^{stop}_{n+1}$, is obtained as follows: Let $\tilde{\Phi}^{stop}_g(\tilde{z}_g)$ be the feasible set of choices for expected cash on hand next period, $\tilde{x}'_e$, when stopping has been chosen:

$$\tilde{\Phi}^{stop}_g(\tilde{z}_g) = \left\{ \tilde{x}'_g \in \tilde{\mathbb{Z}}_{gx} : \frac{\theta_0}{1+\alpha} \leq \tilde{x}'_g \leq \frac{(1+r)\left(\tilde{x}_g - \omega_1\theta_0 - \omega_2(\theta_0 - \tilde{\Theta}_g)\right) + \theta_0}{1+\alpha} \right\}$$

Then for all $\tilde{z}_g \in \tilde{\mathbb{Z}}_g$ where $\tilde{\Phi}^{stop}_g(\tilde{z}_g) \neq \emptyset$:

$$\tilde{v}^{stop}_{n+1}(\tilde{z}_g) = \max_{\tilde{x}'_e \in \tilde{\Phi}^{stop}_g(\tilde{z}_g)} \left\{ \tilde{u}(\tilde{c}') + \beta \sum_{j=0}^{n_y-1} \tilde{v}_n\left(\tilde{\Theta}', \min\left\{\tilde{x}'_{low} + j\cdot g_x, \tilde{x}^{max}\right\}\right) \times p\left(\tilde{x}'_{low} + j\cdot g_x \,\middle|\, \tilde{z}_g, \tilde{\Theta}', \tilde{x}'_e\right)\right\}$$

where $\tilde{\Theta}' = (1+\alpha)^{-1}\theta_0$, and $\tilde{x}'_{low}$ and $p(\tilde{x}'_{low} + j\cdot g_x \mid \tilde{z}_g, \tilde{\Theta}', \tilde{x}'_e)$ are as defined above. Savings and consumption can in this case be derived as: $\tilde{s}' = (1+r)^{-1}(\tilde{x}'_e - (1+\alpha)^{-1}\theta_0)$ and $\tilde{c}' = (1+\alpha)^{-1}(\tilde{x}_g - \omega_1\theta_0 - \omega_2(\theta_0 - \tilde{\Theta}_g)) - \tilde{s}'$.

Proposition 9 can be used to speed up the algorithm when $\omega_2 = 0$. The iterations converge due to the contraction property and will be stopped when the convergence measure in (19) becomes less than some pre-specified level.

Due to the calculation of expectations, the algorithm works somewhat slower than in the deterministic case. Time series simulations can be produced from the obtained policy functions by use of a random number generator to simulate the shocks to income.

5.2 Truncation Problems

As mentioned above, the values of $\tilde{s}^{max}$, and thus $\tilde{x}^{max}$, at which Proposition 6 applies might be too large to be of any practical relevance. However, Proposition 6 simply gives sufficient values of (normalised) savings at which an increase in savings will never take place. The actual level of $\tilde{s}$ where this starts to be the case is likely to be somewhat smaller.

Thus, a truncation of the cash grid is chosen. In general, such truncations may affect the solution, in particular close to the truncation limit. Typical results are sub-estimations of the value function and inaccurate estimates of the optimal policies. However, as long as the probabilities of income realisations implying optimal policy choices outside the truncation limit are small, the truncation will not significantly affect the results well within the limit.

Alternatively, the truncation point could be set close to the maximum level of savings implied by Proposition 6, and a very coarse grid could then be used in those regions where cash on hand is much higher than investment cost. Yet another approach would be to approximate the value function and the policy functions in states outside a limited state space by using simple functional approximations. Both of these approaches will imply increased computation time as they increase the complexity of the maximisation step. To get an idea of the potential gain of implementing one of these approaches, the optimisation problem was solved twice, with the truncation being five in the first run and six in the second. In both cases, the parameter values from Table I were used, together with $\sigma = 0.3$. Evaluating the outcome of these optimisations, it was found that the increase in the value function obtained by increasing the truncation level was nowhere above 0.1%, even at points close to the truncation limit. Likewise, the policy function was hardly affected at all: Only in high technology states very close to the truncation limit ($x_{ij} > 4.5$) did consumption decrease by more than 1%, and never by more than 10%. And additional increases in the truncation limit will have a decreasing effect on the estimations. Given these very limited effects, it was chosen to apply the simple truncation approach. However, a higher η implies a higher precautionary saving motive and hence the truncation limit is increased with η, cf. Table IV.

TABLE IV

TRUNCATION POINTS

η	0.5	0.95	1.05	1.5
$\tilde{x}^{max}$	6	7	8	8

5.3 Simulation Results

The same set of benchmark parameter values is used in the stochastic simulations, together with $\mu = 1$.

Figure 9 presents consumption functions for the case of $\sigma = 0.1$. Compared to the deterministic case, the picture is less sawtoothed. Spikes in the consumption functions are only observed when the level of installed technology is close to, or lower than, the level where reinvestments are undertaken.

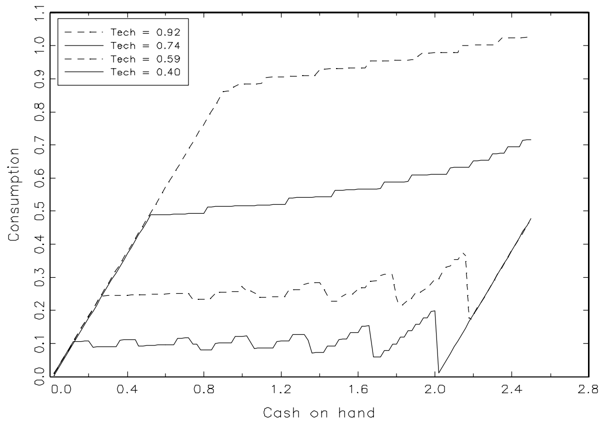

Figure 9: Consumption, $\eta = 0.5$ and $\sigma = 0.1$

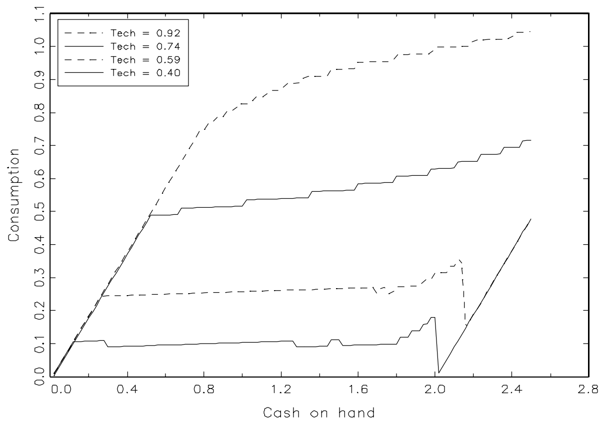

Figure 10: Consumption, $\eta = 0.5$ and $\sigma = 0.3$

No spikes are found at the higher levels of installed technology, where the time to investment is longer.

Increasing the variance obviously diminishes the number of spikes, cf. Figure 10. Uncertainty seems to stabilise consumption. In case of low or no uncertainty, the agent is more capable of planning his future investments. This implies that he might respond to, say, a positive shock by decreasing consumption since this will decrease the time to the next investment. When uncertainty is high, he will not be able to plan with the same certainty, thus causing him to opt for a stable consumption pattern instead.

The sawtoothed picture can also be partly eliminated by increasing η instead of σ. This is visualised in Figure 11. However, the reason is now a

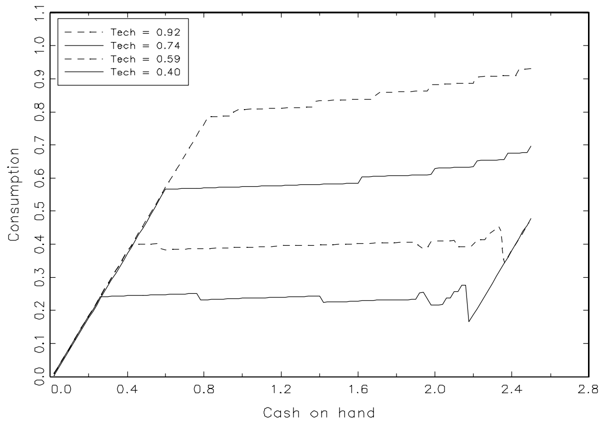

Figure 11: Consumption, $\eta = 1.5$ and $\sigma = 0.1$

different one. When the intertemporal elasticity of substitution is low, η is high, the agent prefers a stable consumption pattern. Even though the small variance in income allows him to adjust his investment and consumption plans in response to shocks, he is not willing to do so and typically reacts by adjusting his savings instead. This is also reflected in the relative flatness of the consumption function compared to the case of $\eta = 0.5$. As an example, consider the case of $\tilde{\Theta} = 0.74$ in Figure 11. The consumption function is virtually horizontal between $\tilde{x} = 0.5$ and $\tilde{x} = 1.6$. With high η, savings tend to be more volatile, whereas consumption will be more volatile at low η.

By further increasing the variance in the case of $\eta = 1.5$, the consumption functions in Figure 12 are obtained. Now, virtually no spikes are left. Thus, a combination of highly variable income and a low intertemporal elasticity of substitution almost completely removes the sawtoothed pattern of consumption.

For both $\eta = 0.5$ and $\eta = 1.5$, increasing uncertainty does not seem to influence the policy functions dramatically, especially not for relatively high levels of installed technology. Thus, in accordance with the literature, the precautionary motive for saving appears to be small. One reason behind this result might be that the amount saved for investment serves a dual role as both buffer-stock saving and investment saving. Thus, the need to hold precautionary balances in excess of savings for investments is small.[14]

[14]This might also be used to explain the small role of precautionary saving in wealth accumulation found by Lusardi (1998) in US data. If agents are credit constrained and save to finance indivisible consumption durables, the savings may serve a precuationary purpose as well. Agents do not hold additional precautionary balances, but instead postpone the purchase of durable goods in case of negative income shocks.

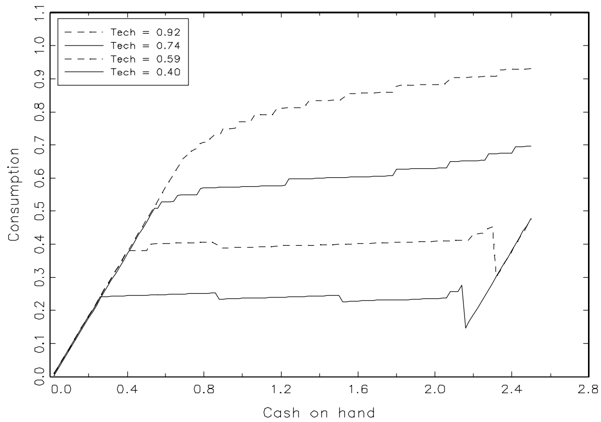

Figure 12: Consumption, $\eta = 1.5$ and $\sigma = 0.3$

When comparing Figures 9 and 10 with 11 and 12, it is also evident that an agent with a higher value of η consumes less of the average income at high levels of technology and more of the average income at low levels of technology. This is again due to the lower intertemporal elasticity of substitution. It causes the agent to save relatively more when income is high, i.e. when technology is high, and relatively less when income is low, i.e. when technology is low. The effect is most pronounced at $\tilde{\Theta} = 0.4$, where consumption is more than twice as high for $\eta = 1.5$ as for $\eta = 0.5$. In these states, technology is unusually low for an agent still opting for a future investment. Normally, there would have been a chance to re-invest before technology had deteriorated to this level. Hence such an agent is on a slow and uncertain course back towards an improved technology. Uncertainty and the preference for a constant consumption level over time imply that an agent with a higher η does not force himself forward as much as an agent with a lower η.

One common characteristic of the presented policy functions is their relative flatness, which implies that consumption does not respond much to income shocks. As mentioned above, the flatness depends both on the elasticity of intertemporal substitution and the uncertainty of income. Furthermore, the option to invest makes current consumption less attractive, and causes a lower marginal propensity to consume than in models without the investment option, see e.g. Deaton (1991).

However, for low levels of technology, the consumption functions become more sawtoothed as cash holdings increase to around the cost of investment. At these levels of technology, consumption responds both positively and negatively to variations in income. Thus, the reaction to income shocks in terms

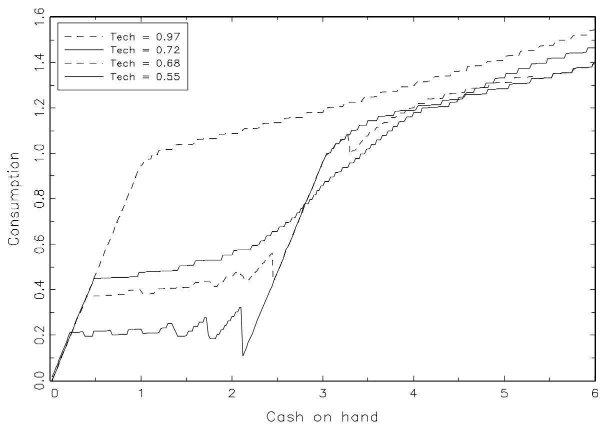

Figure 13: Consumption, $\eta = 0.5$ and $\sigma = 0.1$

of consumption and savings will depend crucially on the timing of shocks. In the beginning of a cycle, only negative shocks will affect consumption. Later in the cycle, the agent will react to shocks by adjusting savings. He is now on the flat part of the policy functions. Towards the end of a cycle, the agent is again likely to adjust his consumption in response to a shock.

As in the deterministic case, a level of cash on hand exists such that up to this level everything is consumed. However, comparing Figure 4 and Figure 10, it is seen that for $\tilde{\Theta} = 0.92$ this level of cash is highest in the deterministic case. This is to be expected due to the precautionary motive in the stochastic case. However, for lower levels of technology, the picture is reversed. The level of cash at which everything is consumed is highest in the stochastic case. This seemingly counter-intuitive result is explained by the investment motive. In the stochastic case, the ability to plan future investments is limited, and hence the investment motive matters less at low levels of cash, leaving only the precautionary motive. In the deterministic case, future investment dates are known and cause the agent to forego current consumption in favour of faster investments and higher future consumption.

Figure 13 is included to yield a more complete picture of the consumption functions for the case of $\eta = 0.5$ and $\sigma = 0.1$. As cash on hand increases, consumption at lower levels of technology "catches up" and tends to "overshoot" consumption at higher levels of technology, even though cash is used partly to finance an investment in the present period when current technology is low. The overshooting is due to the fact that the expected horizon to the *next* investment is then longer at low levels of current technology.

The catching up effect is also visible when comparing $\tilde{\Theta} = 0.72$ with $\tilde{\Theta} = 0.97$ in Figure 13. Whereas the consumption function for $\tilde{\Theta} = 0.97$ exhibits

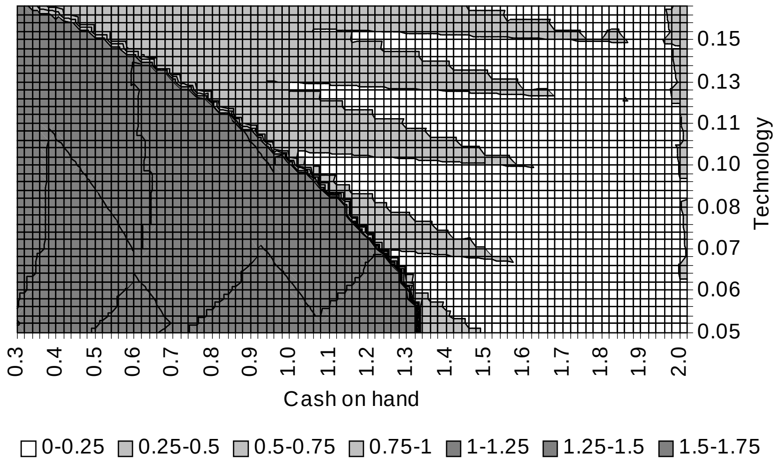

Figure 14: Poverty trap (the legend reports consumption relative to expected income next period including interest on savings), $\eta = 0.95$ and $\sigma = 0.3$

the standard concave shape, well known from the model in Deaton (1991), the corresponding function for $\tilde{\Theta} = 0.72$ has a strictly convex part around $\tilde{x} = 2$. This reflects how the importance attached to a future investment is shifted towards current consumption as the financing of the investment has been secured.

In the above consumption functions, technology levels are too high for the poverty trap to appear. Figure 14 illustrates the poverty trap of a stochastic problem. The poverty trap consists of states where the agent does not find it worthwhile to save for a future investment. It is illustrated in the Figure by those states where the peasant finds it optimal to consume more than his expected income for the next period, including interest on savings. From such states, the agent plans voluntarily to run down his savings and never invest. Note that the agent can be brought into the poverty trap by one or more negative shocks to income and likewise be lifted out of the trap by one or more positive shocks. This is, however, only likely to happen for states rather close to the rim of the dark area in Figure 14, where consumption is close to expected income next period. Sensitivity analyses revealed that there was virtually no poverty trap for low values of η, e.g. $\eta = 0.5$, but the trap increased strongly with increases in η, reflecting that as the coefficient of intertemporal substitution decreases, the willingness to forego consumption now in favour of future gains decreases. Increasing the level of uncertainty had only a marginal effect on the size of the poverty trap, but it definitely increases the probability of being lifted in and out of the poverty trap.

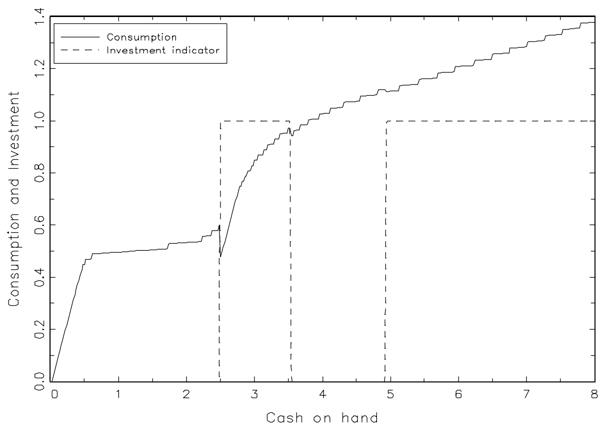

Figure 15: Consumption and investment, $\eta = 1.5$, $\sigma = 0.3$, and $\tilde{\Theta} = 0.68$

Note that an agent standing at the rim of the poverty trap expects the same level of cash in the next period. Since his technology has at the same time declined, he does not expect to escape the trap. In Figure 14 he will expect to enter the dark area. However, by consuming less than the life cycle motive would dictate him in a model without savings, he keeps his investment option alive in case of a positive income shock. Thus, the mere chance, though it might be small, of high income realisations and future investments has a positive effect on savings. This is also true for agents right outside the dark area of Figure 14.

As in the deterministic case, there may exist such "windows", cf. Figure 15, where it is more favourable for an agent to postpone investment and instead enjoy a rather large consumption level now and in the nearest future. In fact, an agent within the window of Figure 15 with $\tilde{x} \geq 2$ expects to invest within one or two periods. As stressed in the discussion of the deterministic case, the result is only present for a small subset of the state space, and it reflects that the decision whether to invest now, i.e. to stop, or to wait, i.e. to continue to hold the option, is a very tight one at these points.

The implications of uncertainty in this model can be analysed in at least two ways. First, obtained policy functions for consumption, saving, and investment can be compared for different parameter values, as was done above. Secondly, simulations can be compared under different kinds of uncertainty by using either single simulations or some average across agents or periods.

Even though policy functions might be identical for different levels of uncertainty, simulations are likely to lead to different realised behaviour. Agents face shocks of different magnitude and will therefore end up taking different routes in the state space. This might influence realised average

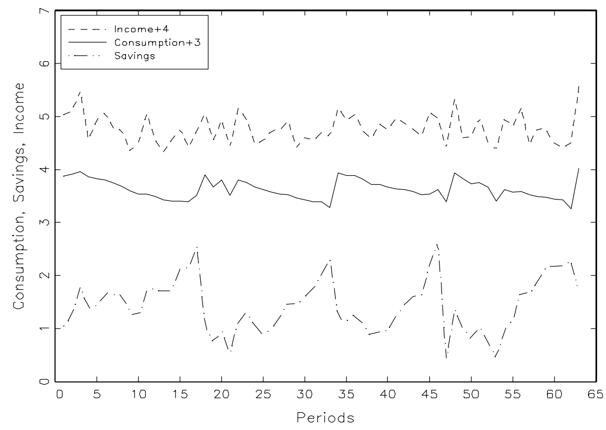

Figure 16: Simulations of $\tilde{y}$, $\tilde{c}$, and $\tilde{s}$, $\eta = 0.5$ and $\sigma = 0.3$

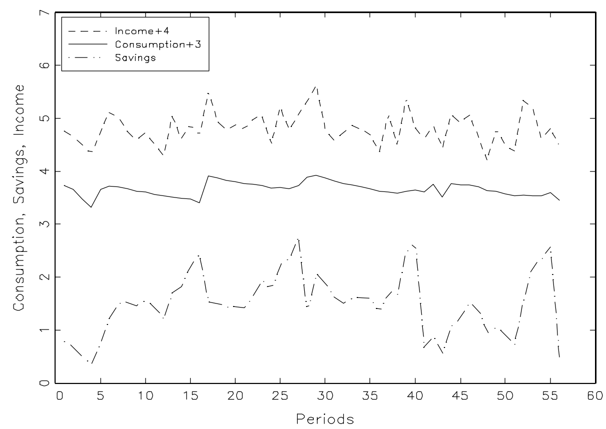

Figure 17: Simulations of $\tilde{y}$, $\tilde{c}$, and $\tilde{s}$, $\eta = 1.5$ and $\sigma = 0.3$

consumption, the length of the investment cycle, etc. Even though shocks average out, their dynamic effects are not likely to do so.

Analysing uncertainty by use of only one of the approaches mentioned is likely to yield an incomplete picture. A priori, it cannot be precluded that the two approaches will yield different conclusions regarding the implications of uncertainty. Thus, for the sake of completeness, the second approach will be considered below.

Figures 16 and 17 present simulated paths of (normalised) income, consumption, and savings. The "kinked" nature of the corresponding policy functions in Figures 10 and 12 implies that for the most relevant cash on hand levels, only negative shocks to income will have a significant effect on consumption. Positive shocks will instead lead to higher savings and faster investment. This asymmetry is illustrated by the drops in consumption in Figure 16 around periods 21 and 53, and around period 43 in Figure 17. At the cycle level, a much smoother consumption pattern is observed for $\eta = 1.5$ than for $\eta = 0.5$. This in spite of identical variance of income.

Table V contains average statistics for different combinations of parameter values. $\bar{p}$ and $\bar{c}$ are defined as in the deterministic case, now just summing over a much larger number of observations. $\bar{s}_{+1}$ is the average level of normalised savings held in the period following an investment.

TABLE V

SENSITIVITY OF PERIOD LENGTH, CONSUMPTION, AND SAVING WITH RESPECT TO σ AND r

σ	0	0	0	0.2	0.2	0.2
r	0.03	0.05	0.07	0.03	0.05	0.07
$\bar{p}$	14	14	15	15.11	14.74	14.56
$\bar{c}$	0.79	0.81	0.85	0.83	0.85	0.88
$\bar{s}_{+1}$	0.045	0.083	0.157	0.128	0.175	0.222

Note: η=0.95 and μ=1, remaining parameter values as in Table I. $\bar{p}$, $\bar{c}$, and $\bar{s}_{+1}$ as defined in text.

The Table shows that as the interest rate increases, average consumption, $\bar{c}$, increases. This is due to the extra income from holding savings. However, the effect on the period length is less clear. In the deterministic case, higher interest rates imply that holding savings become a relatively better tool for generating income, thereby increasing the period length. In the stochastic case, the higher interest rates allow the agent to replace his strategy of precautionary saving in part with a strategy of precautionary investment.

More surprisingly, higher uncertainty has a positive effect on average consumption, whereas the effect on the period length depends crucially on the interest rate. The explanation has to be found in the fact that under higher levels of uncertainty, the agent is saving relatively more in the first periods of a cycle, as indicated by the last row in Table V, thereby generating a higher stock of savings. This allows for both faster investment and higher average consumption as r is increased.

A series of solutions was obtained for various levels of growth in technology, i.e. for various levels of α. The results are reported in Table VI. Evidently, the average investment cycle, $\bar{p}$, decreases as growth in technology increases. The explanation is two-sided. Technology growth makes investing more attractive, but at the same time also more expensive. The effect of the latter is very evident when evaluating the extent of the poverty trap in the

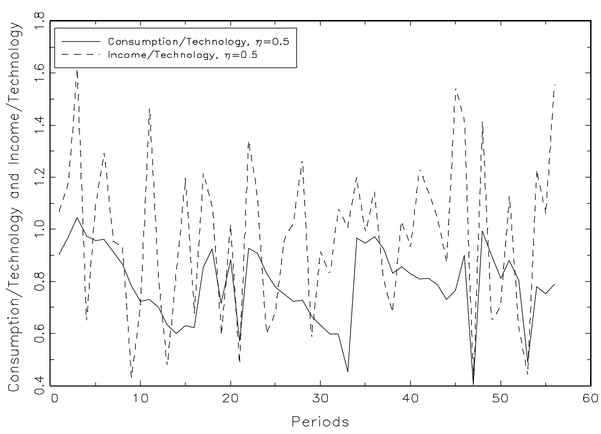

Figure 18: Simulations of y/Θ and c/Θ, $\eta = 0.5$ and $\sigma = 0.3$

model. For agents with a poor technology, increased technology growth only implies that the cost of investment is sooner beyond their reach. Thus, for the peasant ever to be able to invest, either his initial savings or technology must be higher. A shorter investment cycle requires a faster build up of savings. This is reflected in $\bar{c}$. Note that the lower average normalised consumption does not imply a lower non-normalised consumption. Faster technology growth and more frequent investments will imply higher average income.

TABLE VI

SENSITIVITY OF PERIOD LENGTH AND CONSUMPTION WITH RESPECT TO α

α	0.03	0.04	0.05	0.06
$\bar{p}$	14.74	12.22	10.71	9.65
$\bar{c}$	0.85	0.77	0.72	0.67

Note: η=0.95, μ=1, and σ=0.2. Remaining parameter values as in Table I. $\bar{p}$ and $\bar{c}$ as defined in text.

Figures 18 and 19 present simulated paths of $c/\Theta = \tilde{c}/\tilde{\Theta}$ and $y/\Theta = \tilde{y}/\tilde{\Theta}$ for $\eta = 0.5$ and $\eta = 1.5$. Clearly, consumption is somewhat smoother than income, and smoothest when η is high. As mentioned above, consumption is asymmetric in the sense that mostly negative shocks to income are transmitted to consumption. This happens because savings are used to smooth out positive shocks. Thus, co-movement of income and consumption is primarily observed when income is low. However, positive shocks to income might affect consumption towards the end of a cycle.

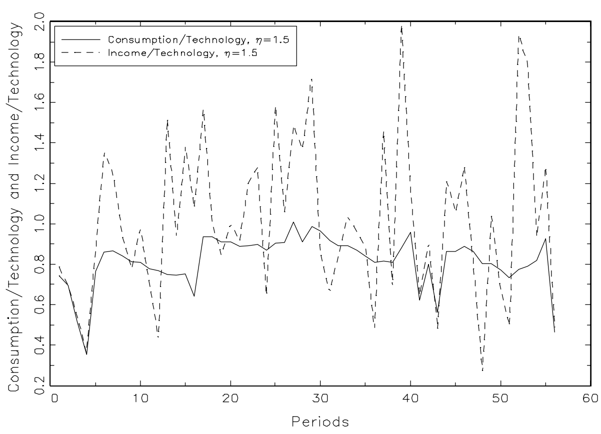

Figure 19: Simulations of y/Θ and c/Θ, $\eta = 1.5$ and $\sigma = 0.3$

Table VII presents the standard deviation of c/Θ for various combinations of σ and η. It appears, that the standard deviation of detrended consumption decreases as η is increased and σ decreased. More interestingly, the increases in standard deviation as σ increases are relatively small, reflecting the fact the policy functions get smoother as σ is increased, cf. the discussion above. Thus, the increase in the variance of consumption is less than would perhaps be expected a priori.

TABLE VII

STANDARD DEVIATION OF INCOME AND CONSUMPTION

$\eta \setminus \sigma$	0.1	0.2	0.3
0.5	0.12	0.13	0.15
0.95	0.07	0.09	0.13
1.05	0.07	0.09	0.11
1.5	0.05	0.07	0.10

Note: μ=1, remaining parameter values as in Table I.

5.4 Alternative Solution Methods

The numerical solution approach used in this paper is a rather simple and, one may say, primitive one. Performing the maximisation step of the algorithm in a number of states, varying between 50,000 and 80,000 in each iteration, assures a very good approximation of the value and policy functions, but is somewhat cumbersome in terms of computation time. A number of alternative methods are often recommended, see e.g. Judd (1998). These include functional approximations of the value function using polynomials, Fourier approximations, or neural networks. Using such approximations, it is often possible to reduce the number of maximisation steps performed and still generate good solutions. They do, however, require fairly well-behaved value functions, i.e. value functions should be smooth and preferably concave. The less well behaved the value functions are, the more parameters must be used in any sort of approximation. This increases computation time and reduces accuracy.

In the initial phases of this work, polynomial representations of the value function were used. While convergence of the polynomials was rather fast, the approximation turned out to be bad. It became clear that the value function in this case is sometimes discontinuous, and never well behaved, in the regions of interest. As a result, this approach was abandoned.

Another alternative to the present value function iteration is to iterate on the policy function by use of a state transition matrix and a payoff function. This is in many cases a much more rapid approach, and it typically generates accurate solutions. However, to be able to illustrate and accurately catch the effect of discontinuities in the value and policy functions, a very fine grid, i.e. a large number of states, is needed. This results in a state transition matrix of gross dimensions between 2,500,000,000 and 6,400,000,000 cells. Because the problem is stochastic, this matrix has non-zero entries in a large part of these cells. Hence it is difficult to condense the matrix significantly. Furthermore, it is likely to be computational cumbersome because the size of the structure forces the computer to use part of the hard disk as memory. This will also be the case during the iterations. Using the hard disk as memory greatly reduces the speed of calculations. Hence the benefits of policy iteration become questionable.

6 Discussion

This section contains a discussion of some of the assumptions in the model and a comparison of the model and its implications with related models in the literature. It also provides several suggestions for future research.

6.1 A Word on Technology and Savings

The peasant considered in this model will in general not hold savings in a bank account or in terms of financial assets. Instead, he will accumulate liquid wealth in assets like cattle, pigs, goats, or other assets easily managed and converted into cash. Thus, the interest rate r in the paper reflects the pay-off from wealth allocated to this use. Note that with this kind of saving there is an implicit replacement problem, which it is assumed that the peasant solves optimally, i.e. the interest rate r is the interest earned on, say, cattle for meat production when breeding and slaughtering is performed optimally and the cash realised is reinvested in the same activity or portfolio of activities.[15]

The development of technology in which the peasant can invest is modelled in a very simple fashion as a single exponential process. As crude as this formulation is, it can still capture essential parts and dynamics of reality. One could argue that a third-world peasant would simply not have the know-how to make use of the newest technology, and hence will always need to buy an older and less productive technology than the newest available. This criticism, however, relies on a much too rigid interpretation of the technology process. A more flexible and relevant interpretation is that the technology process represents the technology that the peasant can buy given his current capacity.[16] As time goes by, he learns more and more from using his own technology and from observing others with similar and better technologies. Therefore, the technology that he potentially can make use of becomes more and more advanced and profitable. The cost function, however, reflects that there will still be a cost related to buying and installing the new technology; a cost which may depend on the difference between currently installed and new technology.

The fact that the peasant cannot buy technologies at a lower level than the maximum is of course a simplification. A natural extension of the model would be to allow the agent to acquire any technology level between Θ_t and θ_t. However, in some sense it seems reasonable to assume that not all intermediate levels of technology are available, or at least not relevant.

Another critique could be that the model works with only one type of technology. Again, this is a very rigid interpretation. In fact, the only implication of the model in that respect is that the investment adds a new technology component to the existing stock of capital; a component which allows for a certain productivity increase. Examples are in plenty: An improved irrigation system, equipment for soil preparation or weed control, genetically improved seeds or animals, improved transportation vehicles which improves his contact to markets, and many other possible investments.

[15]A similar approach is used in Fafchamps and Pender (1997), where cash (or liquid wealth) enters the current income function additively.

[16]This also implies that the technology process is in some sense "agent-specific" and can be interpreted as a reduced form of a more structural setup. It could thus be partly endogenised in a richer (and more complicated) model.

6.2 The Savings-Growth Nexus

The empirical literature generally supports a (weak) positive relationship between saving rates and growth rates at the aggregate level, see e.g. Gersovitz (1988) or Deaton (1990). Neither the standard life-cycle theory nor the buffer-stock models of consumption are able to explain this relationship at the microeconomic level.

In a standard life-cycle model of consumption, growth in income at the individual level creates a motive for dissaving (the life-cycle motive), and thus cannot explain the empirical findings. At the aggregate level, these can be explained if growth takes place *between*, instead of *within*, generations, thereby causing the younger generations to save more.[17]

Compared to the life-cycle model, the buffer stock models in, e.g., Deaton (1991) and Carroll (1997) introduce an extra motive for saving, namely the precautionary motive, arising from the uncertain income in combination with convex marginal utility and/or credit constraints. Deaton (1991) analyses consumption smoothing in the case of non-stationary (growing) income. He finds that savings tend to collapse at zero and that no accumulation takes place during booms. The motive for dissaving arising from consumption smoothing tends to outweigh the need for precautionary saving arising from uncertainty in income. However, the model in Deaton (1991) does not include production and thus lacks an investment motive for saving. The causality runs solely from growth to saving.

The model in this paper incorporates the accumulation motive explicitly and thereby introduces a causality from savings to investments; a link which is able to generate a positive level of savings even when growth in income is expected. The fact that investments have to be financed out of savings creates a motive for saving which is counteracted by the motive for dissaving arising from the growth in income over time. Actually, the model is consistent with the empirical findings; an increase in α (the growth rate) is likely to be associated with an increase in the average propensity to save.

Thus, by including the accumulation motive in addition to the life-cycle and precautionary motives, the model of this paper is believed to yield a richer model of saving (and consumption) behaviour at the microeconomic level than the previous consumption models. The implications of the three different saving motives are clearly visible in the obtained results. First, the accumulation motive is visible in the increasing average balances of savings held during an investment cycle. Secondly, the life cycle motive is reflected in the convex shapes of savings within a cycle. Due to impatience, saving is primarily undertaken in the periods immediately before an investment.

[17] See Deaton (1992) for a comprehensive treatment of this aspect of the life-cycle theory.

This yields a cyclical pattern of both consumption and savings. Thirdly, the precautionary motive works to yield a positive level of savings at the end of a cycle, as well as higher initial savings at the beginning of a cycle when income uncertainty is high.

Note the role played by the credit constraint. It never ceases to affect the behaviour of agents, even though income is constantly growing. Actually, the growth causes the credit constraint to bind, since the impatience of the agent implies that he wishes to spend future income today but is prevented from doing so by the credit constraint.

6.3 The Role of Uncertainty

It is instructive to compare the role played by uncertainty in the present model with that of uncertainty in "pure" consumption models and "pure" investment models.

In the literature on investment under uncertainty, as developed by Dixit and Pindyck (1994) and applied to less developed countries by e.g. Servén (1997), it is emphasised throughout that uncertainty works to decrease investment activity when investments are irreversible. The argument is the following: When investments are impossible to reverse and the (expected) value of an investment changes over time, there exists a "value of waiting". The cost in terms of foregone current income from the investment is typically outweighed by the value from being able to avoid projects that turn out to be unprofitable. In addition, there might be an expected increase in the value of the investment. Thus, "waiting" reduces uncertainty and increases expected pay-off from the investment option.

However, the above set-up does not fully capture the importance of uncertainty to an agricultural household. Farmers are indeed likely to face different investment options, e.g. machinery, education, new production methods, etc., which are (partly) irreversible. And the values of these investments are also likely to change over time, where "value" captures the net present value arising from their addition to the productive capacity of the farm, i.e. to the human and/or physical capital stock. But this is not the whole story – there is an additional source of uncertainty. Farmers have a current income which is highly stochastic. And waiting is not likely to reduce this type of uncertainty; an uncertainty which is caused by weather, prices, or other factors outside the control of the farmer. However, investments might affect the distribution of current income, and thus might serve to *reduce* this uncertainty; if not in a strict variance sense, then by reducing the risk of low outcomes, which are particularly harmful in the case of credit constraints. This aspect is not captured by the pure investment models. They typically assume risk neutrality and thus attach no value to a reduction of this sort of risk. But it is indeed a relevant aspect in a model of an agricultural household.

In dynamic consumption models, on the other hand, the role of uncertainty is basically to cause buffer-stock saving when precautionary motives are present. Thus, only ex-post mechanisms are used to smooth consumption. The agent has no option to affect the distribution of income. This seems a rather unreasonable assumption in an agricultural set-up.

Contrary to this, the option to invest in the present model might be considered as an ex-ante mechanism to smooth consumption. Hence the option introduces the risk reducing aspect of investments often neglected in the investment literature.

Rosenzweig and Wolpin (1993) and Fafchamps and Pender (1997) are probably the most successful among the very few previous attempts to extend the dynamic consumption model with aspects from dynamic investment theory. However, their (empirical) dynamic household models both assume a single, constant investment option, thereby omitting a value of waiting.[18]

Even though there is no uncertainty directly attached to the investment in the model in this paper, there is still a value of waiting arising from the growth in technology.[19] Furthermore, by allowing for repeated investments, the present model extends the set-up of both Rosenzweig and Wolpin (1993) and Fafchamps and Pender (1997), as well as the general set-up in the investment-under-uncertainty literature. This extension is reasonable in the case of agricultural households and provides a fuller dynamic incentive structure than the case of a simple non-repeated investment option.

Servén (1997) documents a negative correlation between uncertainty and investment activity for Africa. He explains it in terms of the "traditional" argument from the investment-under-uncertainty literature that higher uncertainty causes lower investment activity. However, as a result of the above discussion, it seems more likely that credit constraints are to blame for the low investment activity, which then keeps agents in a poverty trap where downside risk is high and financial institutions therefore unwilling to supply credit.

Actually, the standard investment-under-uncertainty set-up cannot explain a long-run link between high uncertainty and low investment activity. It only explains why, in the short run, uncertainty tends to postpone investments. In the long run, uncertainty should actually imply a higher level of

[18]The only value of waiting in these models comes from the emergence of extra cash holdings; a precautionary motive which is typically very small, since the investments tend to reduce the risk of low income realisations.

[19]In case of $\omega_2 = 0$, the net present value of the investment is given by $(1+r)^{-t}\left[(1+\alpha)^t\left(r^{-1}-\omega_1\right)-r^{-1}\right]\Theta$, which is, at least initially, growing in t.

aggregate investment since uncertainty tends to increase the value of investment options. The model of this paper, on the other hand, is consistent with a long-run negative correlation between investment activity and uncertainty.

A more elaborate model could increase the value-of-waiting aspect by introducing uncertainty into the growth process of technology. This could also be achieved by modelling income as an autoregressive process: A low current realisation of income then implies that income is likely to be lower in the near future. This makes current investments relatively less attractive. However, the increased model complexity caused by such extensions would further complicate the interpretation of the dynamic mechanisms, without adding much additional insight. In addition, autocorrelation in income implies an additional dimension in the state space and hence strongly increasing problems of dimensionality in the numerical solutions.

6.4 Policy Implications

Being an attempt to provide a new dynamic framework for modelling agricultural households, the model of this paper is not intended to provide immediate policy implications. However, at least three points deserve comments here.

First, the derived policy functions for consumption and saving seem to have immediate implications for the estimation of household policy functions. If the underlying mechanisms of this model have any empirical relevance, then a simple relationship between consumption and, e.g., wealth, income, and technology cannot be expected to emerge from a microeconometric analysis. The marginal propensity to consume out of income might even be negative in some states. This is contrary to the findings in standard consumption theories such as the life-cycle theory, the permanent income hypothesis, and the buffer-stock model. The highly non-linear and state-dependent policy functions seem to favour direct estimation of the structural model, as done in Rosenzweig and Wolpin (1993) and Fafchamps and Pender (1997). These methods do not rely on consumption being monotone in income and/or cash. This is indeed a relevant concern to politicians aiming at influencing consumption and saving through various policy measures.

Secondly, an implication of the model is that an estimation of utility function parameters should proceed with caution since saving for investment might be confused with precautionary saving. Thus, if the investment motive is not taken into account when estimating a model of consumption and saving, a high value of η and/or a low value of δ (high β) are likely to be obtained.

Thirdly, the model suggests that it is possible to help poor households out of the poverty trap by simple cash transfers. The transfer has to be of a sufficient size, otherwise it is simply consumed. Policies aiming at reducing uncertainty are not likely to have significant effects in the short run, but may speed up investments in the long run. Thus, reducing uncertainty does reduce the poverty trap significantly. Extension of agricultural research and teaching may help to reduce investment cost and thereby increase the speed with which the households can acquire new technology.

6.5 Extensions and Future Research

Since the present paper is to be viewed as a first attempt to provide a more fruitful dynamic framework for the modelling of agricultural households, an important objective is to sketch how further research might proceed from here.

Rosenzweig and Wolpin (1993) argue that the dual role of assets as production factors and as a means of intertemporal consumption smoothing is often ignored in the literature. The present model incorporates two types of assets: a) the installed technology, which can only be used for production; and b) savings, which are used for both consumption smoothing, investment, and production through the generation of interest. Though savings seem to fulfil a dual role, this could be made more explicit by either: i) introducing an extra asset which is more divisible and liquid than technology and contributes directly to production; ii) making the interest earnings a strictly concave function of the amount saved, corresponding to decreasing returns to scale in this asset; or iii) making savings a direct argument in the production function, an approach followed by Fafchamps and Pender (1997).

These extensions should be relatively straightforward. However, it is unclear what additional insight is to be gained from them. Perhaps, individuals will be less willing to smooth consumption by use of savings and will have less incentive to invest. But this effect could perhaps also be captured by a higher rate of interest. Of course, more detailed effects could be obtained with three different assets.

Alternatively, shocks to the productive capacity of the farm could be introduced. Hurricanes, earthquakes, etc. are not uncommon phenomena in many less developed countries, and they might erode the productive base of an agricultural household. A simultaneous shock to capital and income is much worse than a "simple" shock to income. If income depends crucially on a vulnerable stock of capital, shocks have a much more persistent effect. As an example, consider the African Masai with his herd of cattle. A shock to his income will typically be due to a shock to his capital stock, e.g. disease and death of cattle. The effect is of long range and perhaps devastating. A peasant with a major income component from land may also suffer from income variability, but since land does not suddenly disappear, ignoring a few extreme events, his source of wealth is a much less risky one.

The production side of the model can also be extended by a slightly more general production function, perhaps allowing for the use of other inputs in production. However, if these input decisions are not of an explicitly intertemporal nature, the central mechanisms of the present model are likely to go through without major changes.

Furthermore, as discussed above, it would be a natural extension of the model to allow the agent to choose investments from the whole spectrum between his current installed technology level and the prevailing exogenous state of technology.

A cost function which has a higher adjustment cost could also be a viable extension of the model. However, if the normalisation of the model is to go through, it puts some restrictions on feasible functional forms. One alternative could be:

$$\omega\left(\Theta_t, \Theta_{t-1}\right) = \omega_1 \theta_{t-1} + \frac{\omega_2}{\Theta_t}\left(\theta_{t-1} - \Theta_{t-1}\right)^2 \quad if \quad \Theta_t = \theta_{t-1}$$

Estimating the model using the methodology from Rosenzweig and Wolpin (1993) and Fafchamps and Pender (1997) is an obvious empirical extension, though it does require a suitable data set.

7 Conclusion

In this paper, an intertemporal agricultural household model is constructed by combining the standard intertemporal consumption model with extended features from the literature on investment under uncertainty, thereby providing a dynamic alternative to the existing static household models, and extending the theoretical foundations underlying the empirical work on intertemporal household behaviour. The model takes the first step to provide an alternative framework for analysing the intertemporal implications of uncertainty and credit constraints for consumption, saving, and investment behaviour. It is argued, by use of Nicaraguan data, that such intertemporal aspects are indeed relevant to agricultural households in less developed countries.

By considering a normalised version of the model and showing that the state space can be bounded appropriately, it is shown how a solution can be obtained by use of dynamic programming techniques. Minor modifications of the standard Contraction Mapping Theorem are needed to accomplish this step, due to a lack of lower hemicontinuity of the feasible correspondence.

This implies a discontinuous value function for some parameter values. Furthermore, the optimal-stopping nature of the investment decision leaves value function iteration as the only feasible solution method, but can, on the other hand, be used to provide a priori characterisations of the value functions in some special instances. As in the standard dynamic consumption models under credit constraints, marginal utility follows a martingale process which loses its memory whenever assets are run down.

The numerical simulations reveal highly sawtoothed consumption functions. This reflects how changes in wealth affect the investment horizon and thereby current consumption. The sawtoothed pattern is partly removed as either the variance of income is increased or the elasticity of intertemporal substitution decreased, reflecting that the ability to plan future investments and the willingness to respond to intertemporal incentives are decreased, respectively. This implies that increased variation of income is only to a limited extent passed on to consumption. Other results include: i) the precautionary saving motive appears to be small, perhaps reflecting that investment savings serve an additional role as buffer-stock savings; ii) consumption out of cash on hand tends to be higher for low elasticities of intertemporal substitution, since such agents are less susceptible to the benefits of faster investments; iii) consumption functions appear to have much flatter parts than in standard consumption models, reflecting the preference for using investments to increase future consumption at the expense of current consumption; iv) uncertainty can increase the average propensity to consume and speed up investments, since the eagerness to build up precautionary balances yields higher interest income, and because investments themselves might serve a precautionary role; and v) the effects of shocks to income on consumption and saving depend crucially on the timing of the shocks.

By explicitly incorporating three motives for saving, the model is able to yield richer dynamics than standard buffer-stock consumption models with only two motives. The model is able to explain both the existence of savings in case of non-stationary income and a positive correlation between growth rates and saving rates. Hence this model seems better suited for analysing the still puzzling relationship between saving and growth. Furthermore, the role of uncertainty is more appropriately considered than in both the dynamic consumption models with exogenous income, which do not allow for ex-ante actions to affect the uncertainty, and the models of investment under uncertainty, where the role of uncertainty is merely to postpone investments. Finally, by incorporating a true value of waiting and repeated investment options, the model seems more appropriately specified than the few existing (empirical) alternatives in the literature, in addition to providing insights into the mechanisms of these models. Though the model might appear too simple in several respects, it is still a very useful first attempt to simultaneously model the intertemporal consumption and production aspects of an agricultural household.

Even though deriving policy implications is not a specific aim of the present paper, the resulting policy functions do bear immediate relevance for the estimation of consumption functions and parameters of household models. By suggesting several roads for future research, it is furthermore hoped that the methodology laid out in this paper can be a useful point of departure for more theoretical and empirical research on the intertemporal aspects of agricultural household behaviour, and thereby contribute to more successful future policy implementation.

Appendix

Proof of Proposition 1. It must be shown that: i) for any $\tilde{s}_t \geq (1+\alpha)^{-1}(\omega_1+\omega_2)\theta_0$, it will be suboptimal to choose $\tilde{s}_{t+1} > \tilde{s}_t$; and ii) for any $\tilde{s}_t < (1+\alpha)^{-1}(\omega_1+\omega_2)\theta_0$, it will be suboptimal to choose $\tilde{s}_{t+1} > (1+\alpha)^{-1}(\omega_1+\omega_2)\theta_0$.

Ad i). Assume that $\tilde{s}_{t+1} > \tilde{s}_t \geq (1+\alpha)^{-1}(\omega_1+\omega_2)\theta_0$. To show that this choice of $\tilde{s}_{t+1}$ cannot be part of an optimal strategy, the implications of this initial choice will be derived, and it will be shown that there exists an alternative strategy with $\tilde{s}_{t+1} \leq \tilde{s}_t$ which yields strictly higher utility.

$\tilde{s}_{t+1} > \tilde{s}_t$ implies $\tilde{c}_{t+1} < (1+\alpha)^{-1}[(r-\alpha)\tilde{s}_t + y\tilde{\Theta}_t]$, and two future scenarios are then possible: a) $\tilde{s}_{t+j} > 0$ for all $j \geq 2$; and b) $\tilde{s}_{t+j} > 0$ for $j = 2, 3, ..., j'-1$ and $\tilde{s}_{t+j'} = 0$ for some $j' > 1$.

In case of the first scenario, the first order conditions of the problem imply that consumption must satisfy:

$$\tilde{c}_{t+j} = \frac{1}{1+\alpha}\left(\frac{1+r}{1+\delta}\right)^{\frac{1}{\eta}}\tilde{c}_{t+j-1} \leq \frac{1}{1+\alpha}\tilde{c}_{t+j-1} \tag{30}$$

for all $j \geq 1$. However, strictly higher consumption in all periods can then be obtained by choosing $\tilde{s}_{t+j} = \tilde{s}_t$ and $\tilde{\Theta}_{t+j} = (1+\alpha)^{-1}\tilde{\Theta}_{t+j-1}$ for all $j > 0$. In this case:

$$\tilde{c}_{t+j} = \frac{r-\alpha}{1+\alpha}\tilde{s}_t + \frac{y\tilde{\Theta}_t}{(1+\alpha)^j} \geq \frac{\tilde{c}_{t+j-1}}{(1+\alpha)} \tag{31}$$

for $j \geq 2$. This shows the suboptimality of the first scenario.

In case of the second scenario, consumption must satisfy (30) for $j = 2, 3, ..., j'$. In this case, the strategy of $\tilde{s}_{t+j} = \tilde{s}_t$ and $\tilde{\Theta}_{t+j} = (1+\alpha)^{-1}\tilde{\Theta}_{t+j-1}$

for $j = 1,..,j'-1$, and $\tilde{\Theta}_{t+j'} = (1+\alpha)^{-1}\theta_0$ is feasible and can sustain a strictly higher path of consumption, given by (31), for the periods $t+1$ through $t+j'$. Furthermore, the agent will enter period $t+j'+1$ with maximum technology and with no less savings than if he was to follow the strategy of the second scenario. Thus, the second scenario cannot be optimal either. This implies that $\tilde{s}_{t+1} > \tilde{s}_t \geq (1+\alpha)^{-1}(\omega_1+\omega_2)\theta_0$ cannot be part of an optimal strategy.

Ad ii). This step is accomplished in a similar fashion. Choosing $\tilde{s}_{t+1} > (1+\alpha)^{-1}(\omega_1+\omega_2)\theta_0$ implies $\tilde{c}_{t+1} < (1+\alpha)^{-1}[(1+r)\tilde{s}_t + y\tilde{\Theta}_t - (\omega_1+\omega_2)\theta_0]$ and a subsequent consumption pattern given by one of the two scenarios above. However, choosing $\tilde{s}_{t+1} = (1+\alpha)^{-1}(\omega_1+\omega_2)\theta_0$, implying $\tilde{c}_{t+1} = (1+\alpha)^{-1}[(1+r)\tilde{s}_t + y\tilde{\Theta}_t - (\omega_1+\omega_2)\theta_0]$, and otherwise following the alternative strategies from above will again yield strictly higher utility. This completes the second part of the proof. ■

Proof of Proposition 2. Let $\tilde{\mathbb{Z}}$ be a compact set given by (9) for some $\tilde{s}^{max} \geq (1+\alpha)^{-1}(\omega_1+\omega_2)\theta_0$, and let $B(\tilde{\mathbb{Z}})$ be the space of bounded functions $\tilde{v}: \tilde{\mathbb{Z}} \to \mathbb{R}$ with the metric $\rho(\tilde{v}_1,\tilde{v}_2) = \|\tilde{v}_1-\tilde{v}_2\| = \sup_{\tilde{z}\in\tilde{\mathbb{Z}}}|\tilde{v}_1(\tilde{z})-\tilde{v}_2(\tilde{z})|$ for all $\tilde{v}_1,\tilde{v}_2 \in B(\tilde{\mathbb{Z}})$. The structure of the proof is the following: First, it is shown that $B(\tilde{\mathbb{Z}})$ is a complete metric space. Secondly, it is shown that the mapping T defined by (11) is a contraction mapping, taking the space of bounded functions into itself. Together with the completeness of $B(\tilde{\mathbb{Z}})$, this ensures that T has the desired properties. Finally, it is shown that $\tilde{v}'$ corresponds to the unique solution of the sequence problem, $\tilde{v}^*$.

To show that $B(\tilde{\mathbb{Z}})$ is a metric space, it must be shown that the sup norm is a metric on the set of bounded functions. That is, for all $\tilde{v}_1,\tilde{v}_2,\tilde{v}_3 \in B(\tilde{\mathbb{Z}})$, the following conditions must hold:

- $\|\tilde{v}_1-\tilde{v}_2\| \geq 0$, and $\|\tilde{v}_1-\tilde{v}_2\| = 0$ if and only if $\tilde{v}_1=\tilde{v}_2$;
- $\|\tilde{v}_1-\tilde{v}_2\| = \|\tilde{v}_2-\tilde{v}_1\|$; and
- $\|\tilde{v}_1-\tilde{v}_3\| \leq \|\tilde{v}_1-\tilde{v}_2\| + \|\tilde{v}_2-\tilde{v}_3\|$;

The first two conditions are satisfied by inspection, and the third one can be verified by noting that for any $\tilde{z}\in\tilde{\mathbb{Z}}$:

$$\begin{aligned}|\tilde{v}_1(\tilde{z})-\tilde{v}_3(\tilde{z})| &\leq |\tilde{v}_1(\tilde{z})-\tilde{v}_2(\tilde{z})| + |\tilde{v}_2(\tilde{z})-\tilde{v}_3(\tilde{z})| \\ &\leq \|\tilde{v}_1-\tilde{v}_2\| + \|\tilde{v}_2-\tilde{v}_3\| \end{aligned} \quad (32)$$

and since (32) holds for all $\tilde{z}$, it follows that:

$$\|\tilde{v}_1-\tilde{v}_3\| \leq \|\tilde{v}_1-\tilde{v}_2\| + \|\tilde{v}_2-\tilde{v}_3\|$$

Thus, $B(\tilde{\mathbb{Z}})$ is a metric space.

To show that $B(\tilde{\mathbb{Z}})$ is also complete, it must be shown that any Cauchy sequence $\{\tilde{v}_n\}$ in $B(\tilde{\mathbb{Z}})$ converges to an element $\tilde{v}$ in $B(\tilde{\mathbb{Z}})$. This involves three steps: i) to find a candidate function $\tilde{v}$; ii) to show that $\{\tilde{v}_n\}$ converges to $\tilde{v}$ in the sup norm; and iii) to show that $\tilde{v} \in B(\tilde{\mathbb{Z}})$. First, since $\{\tilde{v}_n\}$ is a Cauchy sequence, then for any $\tilde{z} \in \tilde{\mathbb{Z}}$ and for each $\varepsilon > 0$, there exists an M_ε such that for $n, m \geq M_\varepsilon$:

$$|\tilde{v}_n(\tilde{z}) - \tilde{v}_m(\tilde{z})| \leq \|\tilde{v}_n - \tilde{v}_m\| < \varepsilon$$

Thus, the sequence $\{\tilde{v}_n(\tilde{z})\}$ of real numbers is a Cauchy sequence, and by completeness of the real numbers, it has a limit point $\tilde{v}(\tilde{z})$. Let the candidate function be given by these limit points. Secondly, fix $\varepsilon > 0$ and use the fact that $\{\tilde{v}_n\}$ is a Cauchy sequence to choose N_ε such that $\|\tilde{v}_n - \tilde{v}_m\| < \varepsilon/2$ for all $n, m \geq N_\varepsilon$. Now, for arbitrary $\tilde{z} \in \tilde{\mathbb{Z}}$ and all $m \geq n \geq N_\varepsilon$, the following must hold:

$$\begin{aligned} |\tilde{v}_n(\tilde{z}) - \tilde{v}(\tilde{z})| &\leq |\tilde{v}_n(\tilde{z}) - \tilde{v}_m(\tilde{z})| + |\tilde{v}_m(\tilde{z}) - \tilde{v}(\tilde{z})| \\ &\leq \|\tilde{v}_n - \tilde{v}_m\| + |\tilde{v}_m(\tilde{z}) - \tilde{v}(\tilde{z})| \\ &\leq \varepsilon/2 + |\tilde{v}_m(\tilde{z}) - \tilde{v}(\tilde{z})| \end{aligned} \tag{33}$$

And since $\tilde{v}_m(\tilde{z})$ converges to $\tilde{v}(\tilde{z})$, then for each $\tilde{z}$, m can be chosen such that the last term in (33) is less than $\varepsilon/2$. Thus, $\|\tilde{v}_n - \tilde{v}\| \leq \varepsilon$ for all $n \geq N_\varepsilon$. And since the choice of ε was arbitrary, it follows that $\{\tilde{v}_n\}$ converges to $\tilde{v}$ in the sup norm. Thirdly, the boundedness of $\tilde{v}_n$, together with $\|\tilde{v}_n - \tilde{v}\| \leq \varepsilon$ for all $n \geq N_\varepsilon$, implies that $\tilde{v}$ is bounded. Thus, $B(\tilde{\mathbb{Z}})$ is a complete metric space.[20]

Now, since $\tilde{u}$ is bounded on $\tilde{\mathbb{Z}}$, the operator T defined by:

$$(T\tilde{v})(\tilde{z}) = \sup_{\tilde{z}' \in \tilde{\Gamma}(\tilde{z})} [\tilde{u}(\tilde{z}, \tilde{z}') + \beta\tilde{v}(\tilde{z}')]$$

takes the space of bounded functions on $\tilde{\mathbb{Z}}$ into itself: $T : B(\tilde{\mathbb{Z}}) \rightarrow B(\tilde{\mathbb{Z}})$. Furthermore, if $\widetilde{w}, \tilde{v} \in B(\tilde{\mathbb{Z}})$ and $\tilde{v}(\tilde{z}) \geq \widetilde{w}(\tilde{z})$ for all $\tilde{z} \in \tilde{\mathbb{Z}}$, then for any $\tilde{z} \in \tilde{\mathbb{Z}}$:

$$\tilde{u}(\tilde{z}, \tilde{z}') + \beta\tilde{v}(\tilde{z}') \geq \tilde{u}(\tilde{z}, \tilde{z}') + \beta\widetilde{w}(\tilde{z}')$$

for all $\tilde{z}' \in \tilde{\Gamma}(\tilde{z})$. Therefore:

$$\sup_{\tilde{z}' \in \tilde{\Gamma}(\tilde{z})} [\tilde{u}(\tilde{z}, \tilde{z}') + \beta\tilde{v}(\tilde{z}')] \geq \sup_{\tilde{z}' \in \tilde{\Gamma}(\tilde{z})} [\tilde{u}(\tilde{z}, \tilde{z}') + \beta\widetilde{w}(\tilde{z}')]$$

[20] A slightly more general result is proved in Dunford and Schwartz (1958), whereas the proof above resembles the one found in Stokey and Lucas (1989) for the space of bounded and continuous functions.

for any $\tilde{z} \in \tilde{\mathbb{Z}}$. This implies that $(T\tilde{v})(\tilde{z}) \geq (T\tilde{w})(\tilde{z})$ for all $\tilde{z} \in \tilde{\mathbb{Z}}$. Thus, since for any $\tilde{v}_1, \tilde{v}_2 \in B(\tilde{\mathbb{Z}})$:

$$\tilde{v}_1(\tilde{z}) \leq \tilde{v}_2(\tilde{z}) + \|\tilde{v}_1 - \tilde{v}_2\|$$

for all $\tilde{z} \in \tilde{\mathbb{Z}}$, it follows that for all $\tilde{z} \in \tilde{\mathbb{Z}}$:

$$(T\tilde{v}_1)(\tilde{z}) \leq T(\tilde{v}_2 + \|\tilde{v}_1 - \tilde{v}_2\|)(\tilde{z})$$

where $(\tilde{v}_2 + \|\tilde{v}_1 - \tilde{v}_2\|)(\tilde{z})$ is the function defined by: $\tilde{v}_2(\tilde{z}) + \|\tilde{v}_1 - \tilde{v}_2\|$. Thus, for all $\tilde{z} \in \tilde{\mathbb{Z}}$:

$$\begin{aligned}
(T\tilde{v}_1)(\tilde{z}) &\leq \sup_{\tilde{z}' \in \tilde{\Gamma}(\tilde{z})} \left[\tilde{u}(\tilde{z}, \tilde{z}') + \beta(\tilde{v}_2(\tilde{z}') + \|\tilde{v}_1 - \tilde{v}_2\|)\right] \\
&= \sup_{\tilde{z}' \in \tilde{\Gamma}(\tilde{z})} \left[\tilde{u}(\tilde{z}, \tilde{z}') + \beta\tilde{v}_2(\tilde{z}')\right] + \beta\|\tilde{v}_1 - \tilde{v}_2\| \\
&= (T\tilde{v}_2)(\tilde{z}) + \beta\|\tilde{v}_1 - \tilde{v}_2\| \qquad (34)
\end{aligned}$$

And similarly:

$$(T\tilde{v}_2)(\tilde{z}) \leq (T\tilde{v}_1)(\tilde{z}) + \beta\|\tilde{v}_1 - \tilde{v}_2\| \qquad (35)$$

for all $\tilde{z} \in \tilde{\mathbb{Z}}$. Now, (34) and (35) imply that for any $\tilde{v}_1, \tilde{v}_2 \in B(\tilde{\mathbb{Z}})$:

$$\|T\tilde{v}_2 - T\tilde{v}_1\| \leq \beta\|\tilde{v}_1 - \tilde{v}_2\|$$

Thus, T is a contraction mapping with modulus β, taking $B(\tilde{\mathbb{Z}})$ into itself.

Now, since $B(\tilde{\mathbb{Z}})$ is a complete metric space, it follows from the Contraction Mapping Theorem, see Kolmogorov and Fomin (1970) or Stokey and Lucas (1989), that T has a unique fixed point, $\tilde{v}' \in B(\tilde{\mathbb{Z}})$, and that for any $\tilde{v}_0 \in B(\tilde{\mathbb{Z}})$:

$$\|T^n\tilde{v}_0 - \tilde{v}'\| \leq \beta^n\|\tilde{v}_0 - \tilde{v}'\| \quad , \quad n = 0, 1, 2, ...$$

Finally, it must be shown that $\tilde{v}'$ corresponds to $\tilde{v}^*$, the unique solution to the sequence problem in (4)-(8).[21] First, any sequence $\{\tilde{z}_t\}_{t=0}^{\infty}$ in $\tilde{\mathbb{Z}}$ is called a *plan*. For any $\tilde{z}_0 \in \tilde{\mathbb{Z}}$, let:

$$\tilde{\Pi}(\tilde{z}_0) = \left\{\{\tilde{z}_t\}_{t=0}^{\infty} \middle| \tilde{z}_{t+1} \in \tilde{\Gamma}(\tilde{z}_t), \quad t = 0, 1, ...\right\}$$

be the set of *feasible plans* from $\tilde{z}_0$. Let $\{\tilde{z}\} = \{\tilde{z}_0, \tilde{z}_1, \tilde{z}_2, ...\}$ denote an element in $\tilde{\Pi}(\tilde{z}_0)$.

[21]The following proof is a standard one. Similar proofs can be found in, e.g., Stokey and Lucas (1989).

The supremum function $\tilde{v}^*$, which must be finite for any $\tilde{z}_0 \in \tilde{\mathbb{Z}}$, is the unique function satisfying:

$$\tilde{v}^*(\tilde{z}_0) \geq \lim_{n\to\infty} \sum_{i=0}^{n-1} \beta^i \tilde{u}(\tilde{z}_i, \tilde{z}_{i+1}), \quad \textit{all} \quad \{\tilde{z}\} \in \tilde{\Pi}(\tilde{z}_0) \tag{36}$$

$$\tilde{v}^*(\tilde{z}_0) \leq \lim_{n\to\infty} \sum_{i=0}^{n-1} \beta^i \tilde{u}(\tilde{z}_i, \tilde{z}_{i+1}) + \varepsilon, \quad \textit{some} \quad \{\tilde{z}\} \in \tilde{\Pi}(\tilde{z}_0) \tag{37}$$

for all $\tilde{z}_0 \in \tilde{\mathbb{Z}}$ and any $\varepsilon > 0$.

If $\tilde{v} : \tilde{\mathbb{Z}} \to \mathbb{R}$, where $|\tilde{v}(\tilde{z})| < \infty$ for all $\tilde{z} \in \tilde{\mathbb{Z}}$, satisfies (10), then for every $\tilde{z} \in \tilde{\mathbb{Z}}$, $\tilde{v}$ satisfies the following properties for all $\tilde{z}' \in \tilde{\Gamma}(\tilde{z})$ and for some $\tilde{z}'' \in \tilde{\Gamma}(\tilde{z})$ and any $\varepsilon > 0$:

$$\tilde{v}(\tilde{z}) \geq \tilde{u}(\tilde{z}, \tilde{z}') + \beta \tilde{v}(\tilde{z}') \tag{38}$$

$$\tilde{v}(\tilde{z}) \leq \tilde{u}(\tilde{z}, \tilde{z}'') + \beta \tilde{v}(\tilde{z}'') + \varepsilon \tag{39}$$

It remains to show that $\tilde{v}$ satisfies (36) and (37). Now, for any $\tilde{z}_0 \in \tilde{\mathbb{Z}}$, the function $\tilde{v}$ satisfies:

$$\begin{aligned} \tilde{v}(\tilde{z}_0) &\geq \tilde{u}(\tilde{z}_0, \tilde{z}_1) + \beta \tilde{v}(\tilde{z}_1) \\ &\geq \sum_{i=0}^{n-1} \beta^i \tilde{u}(\tilde{z}_i, \tilde{z}_{i+1}) + \beta^n \tilde{v}(\tilde{z}_n) \end{aligned}$$

for all $\{\tilde{z}\} \in \tilde{\Pi}(\tilde{z}_0)$. And since $0 < \beta < 1$ and $|\tilde{v}(\tilde{z})| < \infty$ for all $\tilde{z} \in \tilde{\mathbb{Z}}$, the latter term will converge to zero as n approaches infinity. That is:

$$\tilde{v}(\tilde{z}_0) \geq \lim_{n\to\infty} \sum_{i=0}^{n-1} \beta^i \tilde{u}(\tilde{z}_i, \tilde{z}_{i+1})$$

for all $\{\tilde{z}\} \in \tilde{\Pi}(\tilde{z}_0)$. Thus, $\tilde{v}$ satisfies (36).

Since $\tilde{v}$ satisfies (39), then for any $\tilde{z}_0 \in \tilde{\mathbb{Z}}$ and any sequence $\{\mu_t\}_{t=1}^{\infty}$ where $\mu_t > 0$, there exists some $\{\tilde{z}\} \in \tilde{\Pi}(\tilde{z}_0)$ such that:

$$\tilde{v}(\tilde{z}_t) \leq \tilde{u}(\tilde{z}_t, \tilde{z}_{t+1}) + \beta \tilde{v}(\tilde{z}_{t+1}) + \mu_{t+1}, \quad t = 0, 1, \ldots$$

and therefore:

$$\tilde{v}(\tilde{z}_0) \leq \sum_{i=0}^{n-1} \beta^i \tilde{u}(\tilde{z}_i, \tilde{z}_{i+1}) + \beta^n \tilde{v}(\tilde{z}_n) + \left(\mu_1 + \beta\mu_2 + \ldots + \beta^{n-1}\mu_n\right)$$

Now, for any arbitrary $\varepsilon > 0$, let $\mu_t = \frac{1}{2}(1-\beta)\varepsilon$. Then $\sum_{i=0}^{n-1} \beta^i \mu_{i+1} \leq \varepsilon/2$ for all $n \geq 0$, i.e.:

$$\tilde{v}(\tilde{z}_0) \leq \sum_{i=0}^{n-1} \beta^i \tilde{u}(\tilde{z}_i, \tilde{z}_{i+1}) + \beta^n \tilde{v}(\tilde{z}_n) + \frac{\varepsilon}{2}$$

And for sufficiently large n: $\beta^n \tilde{v}(\tilde{z}_n) < \frac{\varepsilon}{2}$. Thus:

$$\tilde{v}(\tilde{z}_0) \leq \sum_{i=0}^{n-1} \beta^i \tilde{u}(\tilde{z}_i, \tilde{z}_{i+1}) + \varepsilon$$

for some sufficiently large n, which means that $\tilde{v}$ satisfies (37). Thus, $\tilde{v}'$ corresponds to $\tilde{v}^*$. This completes the proof of Proposition 2. ■

Proof of Proposition 3. The structure of the proof is the following: First, it is shown that the mapping T takes the space of bounded functions on $\tilde{\mathbb{Z}}_b$ into itself. Secondly, it is proven that the policy correspondence is upper hemicontinuous; a result which is used to show that T takes the space of bounded and *continuous* functions into itself. Finally, it is verified that the Contraction Mapping Theorem applies to T. This yields the desired results.

The mapping T was defined by:

$$(T\tilde{v})(\tilde{z}) = \sup_{\tilde{z}' \in \tilde{\Gamma}(\tilde{z})} [\tilde{u}(\tilde{z}, \tilde{z}') + \beta \tilde{v}(\tilde{z}')] \quad (40)$$

where $\tilde{\Gamma}(\tilde{z})$ is non-empty and compact for any $\tilde{z} \in \tilde{\mathbb{Z}}_b$. First, if $\tilde{s}^{min} > 0$, then, since $r > \alpha$, $\tilde{c}' = (1+\alpha)^{-1}(r-\alpha)\tilde{s}^{min} > 0$ is a feasible strategy for all $\tilde{\Theta} \geq 0$ and $\tilde{s} \geq \tilde{s}^{min}$. Secondly, if $\tilde{\Theta}^{min} > 0$, then $\tilde{c}' = (1+\alpha)^{-1} y \tilde{\Theta}^{min} > 0$ is feasible for all $\tilde{\Theta} \geq \tilde{\Theta}^{min}$ and $\tilde{s} \geq 0$. This implies that the value function is bounded from below. Since it is also bounded from above, then, for any $\tilde{v} \in B(\tilde{\mathbb{Z}}_b)$, the mapping T in (40) yields another bounded function, $T : B(\tilde{\mathbb{Z}}_b) \to B(\tilde{\mathbb{Z}}_b)$. Hence $\tilde{c} = 0$ can never be an optimal choice.

Thus, the proof of Proposition 2 applies in this case as well. However, in the following it will be shown that when $\eta \geq 1$, the mapping T operates on the space of bounded and *continuous* functions, $C(\tilde{\mathbb{Z}}_b)$.

If $\tilde{v} \in C(\tilde{\mathbb{Z}}_b)$, then $\tilde{u}(\tilde{z}, \tilde{z}') + \beta \tilde{v}(\tilde{z}')$ is continuous in $\tilde{z}'$. This implies that the maximum in (40) is attained, and therefore the policy correspondence:

$$\tilde{G}(\tilde{z}) = \left\{ \tilde{z}' \in \tilde{\Gamma}(\tilde{z}) : (T\tilde{v})(\tilde{z}) = \tilde{u}(\tilde{z}, \tilde{z}') + \beta \tilde{v}(\tilde{z}') \right\} \quad (41)$$

is non-empty. Now, since $\tilde{G}(\tilde{z}) \subseteq \tilde{\Gamma}(\tilde{z})$ and $\tilde{\Gamma}(\tilde{z})$ is compact, then $\tilde{G}(\tilde{z})$ must be bounded. And suppose that $z_n' \to z'$ and $\tilde{z}_n' \in \tilde{G}(\tilde{z})$ for all n. This implies that $\tilde{z}'_n \in \tilde{\Gamma}(\tilde{z})$ all n, and since $\tilde{\Gamma}(\tilde{z})$ is closed, then $\tilde{z}' \in \tilde{\Gamma}(\tilde{z})$. Furthermore, since $\tilde{u}(\tilde{z}, \tilde{z}'_n) + \beta\tilde{v}(\tilde{z}'_n)$ is constant for all n, it must equal $\tilde{u}(\tilde{z}, \tilde{z}') + \beta\tilde{v}(\tilde{z}')$ due to continuity. This implies that $\tilde{z}' \in \tilde{G}(\tilde{z})$. Thus, $\tilde{G}(\tilde{z})$ is closed. Now, since $\tilde{G}(\tilde{z})$ is both bounded and closed, it is compact. This holds for all $\tilde{z}$ in $\tilde{\mathbb{Z}}_b$, implying that $\tilde{G}$ is non-empty and compact.

To show that $\tilde{G}$ is also upper hemicontinuous, it must be shown that at any $\tilde{z} \in \tilde{\mathbb{Z}}_b$, for every sequence $\{\tilde{z}_n\}$ converging to $\tilde{z}$, and every sequence $\{\tilde{z}'_n\}$ such that $\tilde{z}'_n \in \tilde{G}(\tilde{z}_n)$ for all n, there exists a subsequence of $\{\tilde{z}'_n\}$, call it $\{\tilde{z}'_{n_k}\}$, which converges to $\tilde{z}' \in \tilde{G}(\tilde{z})$.

First, consider any $\tilde{z} \in \tilde{\mathbb{Z}}_b$ such that $(1+r)\tilde{s} + y\tilde{\Theta} - (1+\alpha)\tilde{s}^{min} \neq \omega_1\theta_0 + \omega_2(\theta_0 - \tilde{\Theta})$. And let $\{\tilde{z}_n\}$ be any sequence converging to $\tilde{z}$. Then choose $\tilde{z}'_n \in \tilde{G}(\tilde{z}_n)$ for all n, i.e. the $\tilde{z}'_n$'s are maximisers. Now, since $\tilde{\Gamma}(\tilde{z})$ is upper hemicontinuous, there exists a subsequence $\{\tilde{z}'_{n_k}\}$ converging to $\tilde{z}'$ with $\tilde{z}' \in \tilde{\Gamma}(\tilde{z})$. In the case of a $\tilde{z}$ where $(1+r)\tilde{s} + y\tilde{\Theta} - (1+\alpha)\tilde{s}^{min} \neq \omega_1\theta_0 + \omega_2(\theta_0 - \tilde{\Theta})$, $\tilde{\Gamma}(\tilde{z})$ is also lower hemicontinuous at $\tilde{z}$. This implies that for any $\tilde{z}'' \in \tilde{\Gamma}(\tilde{z})$, there exists a subsequence $\{\tilde{z}''_{n_k}\}$ such that $\tilde{z}''_{n_k} \to \tilde{z}''$ with $\tilde{z}''_{n_k} \in \tilde{\Gamma}(\tilde{z}_{n_k})$ for all k. Now, since $\tilde{u}(\tilde{z}_{n_k}, \tilde{z}'_{n_k}) + \beta\tilde{v}(\tilde{z}'_{n_k}) \geq \tilde{u}(\tilde{z}_{n_k}, \tilde{z}''_{n_k}) + \beta\tilde{v}(\tilde{z}''_{n_k})$ for all k, then $\tilde{u}(\tilde{z}, \tilde{z}') + \beta\tilde{v}(\tilde{z}') \geq \tilde{u}(\tilde{z}, \tilde{z}'') + \beta\tilde{v}(\tilde{z}'')$ due to continuity of the functions. This holds for all $\tilde{z}'' \in \tilde{\Gamma}(\tilde{z})$, which implies that $\tilde{z}' \in \tilde{G}(\tilde{z})$.

Then consider the case of a $\tilde{z} \in \tilde{\mathbb{Z}}_b$ where $(1+r)\tilde{s} + y\tilde{\Theta} - (1+\alpha)\tilde{s}^{min} = \omega_1\theta_0 + \omega_2(\theta_0 - \tilde{\Theta})$. And let $\tilde{z}^{inv}$ be a feasible choice of $\tilde{z}'$ at $\tilde{z}$ where an investment is made, i.e. $\tilde{z}^{inv} \in \tilde{\Gamma}(\tilde{z})$. But since $\tilde{z}^{inv}$ implies $\tilde{c} = 0$, it follows from above that $\tilde{z}^{inv} \notin \tilde{G}(\tilde{z})$. Thus, without loss of generality, such points can be excluded from the feasible set. Then for all other $\tilde{z}'' \in \tilde{\Gamma}(\tilde{z})$, there exists a subsequence $\{\tilde{z}''_{n_k}\}$ such that $\tilde{z}''_{n_k} \to \tilde{z}''$ with $\tilde{z}''_{n_k} \in \tilde{\Gamma}(\tilde{z}_{n_k})$ for all k. Again, continuity can be used to establish that $\tilde{u}(\tilde{z}, \tilde{z}') + \beta\tilde{v}(\tilde{z}') \geq \tilde{u}(\tilde{z}, \tilde{z}'') + \beta\tilde{v}(\tilde{z}'')$ for all these $\tilde{z}'' \in \tilde{\Gamma}(\tilde{z})$. This implies that $\tilde{z}' \in \tilde{G}(\tilde{z})$. Thus, $\tilde{G}$ is upper hemicontinuous.

The fact that $\tilde{G}$ is upper hemicontinuous can then be used to show that the mapping T yields a continuous function. Consider any $\tilde{z} \in \tilde{\mathbb{Z}}_b$ and any sequence $\{\tilde{z}_n\}$ converging to $\tilde{z}$. It has to be shown that $(T\tilde{v})(\tilde{z}_n) \to (T\tilde{v})(\tilde{z})$. Assume that this is not the case. Thus, there exists $\varepsilon > 0$ and a subsequence $\{\tilde{z}_{n_k}\}$ such that: $|(T\tilde{v})(\tilde{z}_{n_k}) - (T\tilde{v})(\tilde{z})| > \varepsilon$ for all k. Furthermore, there exists $\tilde{z}'_{n_k} \in \tilde{G}(\tilde{z}_{n_k})$ such that $(T\tilde{v})(\tilde{z}_{n_k}) = \tilde{u}(\tilde{z}_{n_k}, \tilde{z}'_{n_k}) + \beta\tilde{v}(\tilde{z}'_{n_k})$ for all k. The fact that $\tilde{G}$ is upper hemicontinuous implies that there exists a subsequence of $\{\tilde{z}'_{n_k}\}$ which converges to $\tilde{z}' \in \tilde{G}(\tilde{z})$. This implies that $(T\tilde{v})(\tilde{z}) = \tilde{u}(\tilde{z}, \tilde{z}') + \beta\tilde{v}(\tilde{z}')$, and continuity of $\tilde{u}(\tilde{z}_{n_k}, \tilde{z}'_{n_k}) + \beta\tilde{v}(\tilde{z}'_{n_k})$ implies that:

$$\begin{aligned}\lim_{n\to\infty}(T\tilde{v})(\tilde{z}_{n_k}) &= \lim_{n\to\infty}\left\{\tilde{u}\left(\tilde{z}_{n_k},\tilde{z}'_{n_k}\right)+\beta\tilde{v}\left(\tilde{z}'_{n_k}\right)\right\} \\ &= \tilde{u}(\tilde{z},\tilde{z}')+\beta\tilde{v}(\tilde{z}') \\ &= (T\tilde{v})(\tilde{z})\end{aligned}$$

which contradicts the initial assumption. Thus, $(T\tilde{v})(\tilde{z})$ must be continuous, and the mapping T therefore takes the space of bounded and continuous functions into itself.

Finally, by the proof of Proposition 2, T is a contraction mapping with modulus β. And since $C(\tilde{\mathbb{Z}}_b)$ is a complete metric space, see Stokey and Lucas (1989), T satisfies the Contraction Mapping Theorem. Thus, T has a unique fixed point, $\tilde{v}' \in C(\tilde{\mathbb{Z}}_b)$, and for any $\tilde{v}_0 \in C(\tilde{\mathbb{Z}}_b)$:

$$\|T^n\tilde{v}_0-\tilde{v}'\| \leq \beta^n \|\tilde{v}_0-\tilde{v}'\| \quad , \quad n=0,1,2...$$

which completes the proof. ■

Proof of Proposition 5. First, when $\omega_2 = 0$, the value of stopping, $\tilde{v}^{stop}(\tilde{x},\tilde{\Theta})$, is constant for all $\tilde{\Theta} \in [\tilde{\Theta}^{min},(1+\alpha)^{-1}\theta_0]$. Secondly, the value of continuing, $\tilde{v}^{cont}(\tilde{x},\tilde{\Theta})$, must be strictly increasing in $\tilde{\Theta}$, since strictly higher $\tilde{\Theta}$ implies strictly higher income next period and thereby strictly higher consumption possibilities. Thus, given $\tilde{x}$, there exists a $\tilde{\Theta}' \in [\tilde{\Theta}^{min},(1+\alpha)^{-1}\theta_0]$ which separates the stopping region from the continuation region, unless continuing or stopping is optimal for all $\tilde{\Theta} \in [\tilde{\Theta}^{min},(1+\alpha)^{-1}\theta_0]$, i.e. case i) or ii) in the Proposition. Furthermore, to show that $\tilde{v}^{cont}(\tilde{x},\tilde{\Theta}') = \tilde{v}^{stop}(\tilde{x},\tilde{\Theta}')$ when neither i) nor ii) applies, it is sufficient to show that $\tilde{v}^{cont}(\tilde{x},\tilde{\Theta})$ is also continuous in $\tilde{\Theta} \in [\tilde{\Theta}^{min},(1+\alpha)^{-1}\theta_0]$.

Continuity of $\tilde{v}^{cont}(\tilde{x},\tilde{\Theta})$ in $\tilde{\Theta} \in [\tilde{\Theta}^{min},(1+\alpha)^{-1}\theta_0]$ requires that for each $\tilde{\Theta}^a \in [\tilde{\Theta}^{min},(1+\alpha)^{-1}\theta_0]$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that $|\tilde{v}^{cont}(\tilde{x},\tilde{\Theta}^a) - \tilde{v}^{cont}(\tilde{x},\tilde{\Theta}^b)| < \varepsilon$ for any $\tilde{\Theta}^b \in [\tilde{\Theta}^{min},(1+\alpha)^{-1}\theta_0]$ such that $|\tilde{\Theta}^a - \tilde{\Theta}^b| < \delta$.

To see that continuity holds, note that strictly positive consumption must be part of an optimal continuation strategy for any $\tilde{\Theta} \in [\tilde{\Theta}^{min},(1+\alpha)^{-1}\theta_0]$ when $\tilde{x} \geq \omega_1\theta_0 + (1+\alpha)\tilde{s}^{min}$, since $\tilde{u}'(\tilde{c}_t) \to \infty$ as $\tilde{c} \to 0$ and positive consumption is feasible without restrictions on future investment options. Now, given $\delta > 0$, consider any $\tilde{\Theta}^a, \tilde{\Theta}^b \in [\tilde{\Theta}^{min},(1+\alpha)^{-1}\theta_0]$ such that $\tilde{\Theta}^a < \tilde{\Theta}^b$ and $|\tilde{\Theta}^a - \tilde{\Theta}^b| < \delta$. When δ is chosen sufficiently small, current consumption under any $\tilde{\Theta}^a$ can be decreased by $r^{-1}\delta y$ compared to consumption under $\tilde{\Theta}^b$. This makes the value function next period no less under $\tilde{\Theta}^a$ than under $\tilde{\Theta}^b$. This is because the extra interest income next period, δy, compensates sufficiently for the income differential due to $\tilde{\Theta}^a$ being smaller than $\tilde{\Theta}^b$. And the current loss in utility can be made arbitrarily small by appropriate choice of δ. Thus, for any $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\left|\tilde{v}^{cont}(\tilde{x},\tilde{\Theta}^a) - \tilde{v}^{cont}(\tilde{x},\tilde{\Theta}^b)\right| < \left|\tilde{u}(\tilde{c}^{cont}(\tilde{x},\tilde{\Theta}^b)) - \tilde{u}(\tilde{c}^{cont}(\tilde{x},\tilde{\Theta}^b) - r^{-1}\delta y)\right| < \varepsilon$$

where $\tilde{c}^{cont}(\tilde{x},\tilde{\Theta}^b)$ is optimal consumption under a continuation strategy given $\tilde{x}$ and $\tilde{\Theta}^b$. This establishes continuity of $\tilde{v}^{cont}(\tilde{x},\tilde{\Theta})$. ■

Proof of Proposition 6. It must be shown that there exists a $\tilde{s}^{high} < \infty$ such that if $\tilde{s}_0 \leq \tilde{s}^{max} \geq \tilde{s}^{high}$, then $\tilde{s}_t \leq \tilde{s}^{max}$ for all $t > 0$. To do this, it is sufficient to show that there exists $\tilde{s}^* < \infty$ such that if $\tilde{s}_t \geq \tilde{s}^*$, then $\tilde{s}_{t+1} \leq \tilde{s}_t$. Then $\tilde{s}^{high}$ can be defined by $\tilde{s}^{high} = (1+\alpha)^{-1}\left[(1+r)\,\tilde{s}^* + \overline{m}\,(1+\alpha)^{-1}\theta_0\right]$. To show that $\tilde{s}^*$ exists, it is sufficient to show that there exists a $\tilde{s}^*$ such that:

$$\tilde{c}_{t+1} \geq \frac{(r-\alpha)\,\tilde{s}_t + \tilde{y}^{max}}{1+\alpha} \tag{42}$$

for all $\tilde{s}_t \geq \tilde{s}^*$, where $\tilde{y}^{max} = \overline{m}\,(1+\alpha)^{-1}\theta_0$.

First, consider a corresponding deterministic problem where the agent is endowed with initial wealth:

$$\tilde{w}_t = (1+\alpha)^{-1}\tilde{x}_t = (1+\alpha)^{-1}(1+r)\,\tilde{s}_t - (r-\alpha)^{-1}(1+r)\,\tilde{\omega}^{max},$$

where $\tilde{\omega}^{max} = (\omega_1+\omega_2)(1+\alpha)^{-1}\theta_0$, the maximum cost of investment. That is, the term $(r-\alpha)^{-1}(1+r)\,\tilde{\omega}^{max}$ is the present value of paying for an investment each period. The agent receives zero income each period and wealth, $\tilde{w}_t$, is divided between consumption and savings. Thus, wealth evolves according to:

$$\tilde{w}_{t+1} = (1+\alpha)^{-1}(1+r)\left[\tilde{w}_t - \tilde{c}_{t+1}\right]$$

In such a model, current consumption is given by:

$$\begin{aligned}
\tilde{c}_{t+1} &= \left(1 - \frac{\beta^{\frac{1}{\eta}}(1+\alpha)^{-\frac{1}{\eta}}(1+r)^{\frac{1}{\eta}}}{(1+\alpha)^{-1}(1+r)}\right)\tilde{w}_t \\
&= \left(1 - \frac{\beta^{\frac{1}{\eta}}(1+\alpha)^{-\frac{1}{\eta}}(1+r)^{\frac{1}{\eta}}}{(1+\alpha)^{-1}(1+r)}\right)\left(\frac{1+r}{1+\alpha}\tilde{s}_t - \frac{1+r}{r-\alpha}\tilde{\omega}^{max}\right) \qquad (43)
\end{aligned}$$

see e.g. Carroll (1997).

Clearly, consumption in the stochastic model will be no less than in this deterministic model since stochastic income is bounded from below by zero, and total investment cost will always be smaller than $(r-\alpha)^{-1}(1+r)\tilde{\omega}^{max}$.

It only remains to show that there exists a value $\tilde{s}^*$ such that the right hand side in (43) exceeds the right hand side in (42) for all values of $\tilde{s}_t \geq \tilde{s}^*$. These values of $\tilde{s}_t$ should thus satisfy:

$$\frac{1+r}{1+\alpha}\tilde{s}_t - \beta^{\frac{1}{\eta}}(1+\alpha)^{-\frac{1}{\eta}}(1+r)^{\frac{1}{\eta}}\tilde{s}_t - \frac{(r-\alpha)\tilde{s}_t}{1+\alpha} \geq K$$

where K is some constant depending on the parameters. Using the fact that $\beta = (1+\delta)^{-1}(1+\alpha)^{1-\eta}$, the inequality simplifies to:

$$\left[1+\alpha-\left(\frac{1+r}{1+\delta}\right)^{\frac{1}{\eta}}\right]\tilde{s}_t \geq K(1+\alpha)$$

which is obviously satisfied for all $\tilde{s}_t$ larger than some $\tilde{s}^*$. ■

Proof of Proposition 7. Let $\tilde{\mathbb{X}} = \tilde{\mathbb{Z}} \times \mathbb{M}$ be the compact space where $\tilde{\mathbb{Z}}$ is given by (9) for some $\tilde{s}^{max}$ satisfying Proposition 6, and let $B(\tilde{\mathbb{X}})$ be the space of bounded functions $\tilde{v} : \tilde{\mathbb{X}} \to \mathbb{R}$ with the metric $\rho(\tilde{v}_1, \tilde{v}_2) = \|\tilde{v}_1 - \tilde{v}_2\| = \sup_{\tilde{x}\in\tilde{\mathbb{X}}} |\tilde{v}_1(\tilde{x}) - \tilde{v}_2(\tilde{x})|$ for all $\tilde{v}_1, \tilde{v}_2 \in B(\tilde{\mathbb{X}})$. Then $B(\tilde{\mathbb{X}})$ is a complete metric space by arguments analogous to those used in the proof of Proposition 2. The rest of the proof is also similar in structure to that of Proposition 2. First, it is shown that the mapping T defined by (24) is a contraction mapping, taking the space of bounded functions into itself. This implies that T is a contraction mapping with the desired properties. Finally, it is argued that $\tilde{v}'$ corresponds to the unique solution of the sequence problem, $\tilde{v}^*$.

Now, since $\tilde{u}$ and $\tilde{v}$ are bounded, the mapping T defined by:

$$(T\tilde{v})(\tilde{z}, m) = \sup_{\tilde{z}'\in\tilde{\Gamma}(\tilde{z},m)}\left[\tilde{u}(\tilde{z}, \tilde{z}', m) + \beta\int_{\underline{m}}^{\overline{m}}\tilde{v}(\tilde{z}', m')\,dF(m')\right]$$

takes $B(\tilde{\mathbb{X}})$ into itself.[22] Furthermore, if $\tilde{v}(\tilde{z}, m) \geq \tilde{w}(\tilde{z}, m)$ for all $(\tilde{z}, m) \in$

[22]Throughout, it is implicitly assumed that the integral is well defined, and that the mapping yields an integrable function. One way to ensure this is to use the approach outlined by Denardo (1976). He defines an operator, instead of an integral, which is the same as the integral if the latter exists, and otherwise substitutes $\tilde{v}$ with the integrable function $\tilde{\omega}$ $(\geq \tilde{v})$ which yields the smallest possible value of the integral. Since such problems of measurability are mostly theoretical curiosities, and of no practical relevance, they are not treated further in this paper. For any reasonably specified value function, the integral is well defined and the mapping will yield another integrable function.

$\tilde{\mathbb{Z}} \times \mathbb{M}$, it follows that:

$$\int_{\underline{m}}^{\overline{m}} \tilde{v}\left(\tilde{z}, m\right) dF\left(m\right) \geq \int_{\underline{m}}^{\overline{m}} \tilde{w}\left(\tilde{z}, m\right) dF\left(m\right)$$

for all $\tilde{z} \in \tilde{\mathbb{Z}}$. And for any $(\tilde{z}, m) \in \tilde{\mathbb{Z}} \times \mathbb{M}$:

$$\tilde{u}\left(\tilde{z}, \tilde{z}', m\right) + \beta \int_{\underline{m}}^{\overline{m}} \tilde{v}\left(\tilde{z}', m'\right) dF\left(m'\right) \geq \tilde{u}\left(\tilde{z}, \tilde{z}', m\right) + \beta \int_{\underline{m}}^{\overline{m}} \tilde{w}\left(\tilde{z}', m'\right) dF\left(m'\right)$$

for all $\tilde{z}' \in \tilde{\Gamma}\left(\tilde{z}, m\right)$. This implies:

$$\sup_{\tilde{z}' \in \tilde{\Gamma}(\tilde{z}, m)} \left[\tilde{u}\left(\tilde{z}, \tilde{z}', m\right) + \beta \int_{\underline{m}}^{\overline{m}} \tilde{v}\left(\tilde{z}', m'\right) dF\left(m'\right)\right] \geq \sup_{\tilde{z}' \in \tilde{\Gamma}(\tilde{z}, m)} \left[\tilde{u}\left(\tilde{z}, \tilde{z}', m\right) + \beta \int_{\underline{m}}^{\overline{m}} \tilde{w}\left(\tilde{z}', m'\right) dF\left(m'\right)\right]$$

which again implies that $\left(T\tilde{v}\right)\left(\tilde{z}, m\right) \geq \left(T\tilde{w}\right)\left(\tilde{z}, m\right)$ for all $(\tilde{z}, m) \in \tilde{\mathbb{Z}} \times \mathbb{M}$. And, since for any $\tilde{v}_1, \tilde{v}_2 \in B(\tilde{\mathbb{X}})$:

$$\tilde{v}_1\left(\tilde{z}, m\right) \leq \tilde{v}_2(\tilde{z}, m) + \|\tilde{v}_1 - \tilde{v}_2\|$$

for all $(\tilde{z}, m) \in \tilde{\mathbb{Z}} \times \mathbb{M}$, it follows that for all $(\tilde{z}, m) \in \tilde{\mathbb{Z}} \times \mathbb{M}$:

$$\left(T\tilde{v}_1\right)\left(\tilde{z}, m\right) \leq T\left(\tilde{v}_2 + \|\tilde{v}_1 - \tilde{v}_2\|\right)\left(\tilde{z}, m\right)$$

where $\left(\tilde{v}_2 + \|\tilde{v}_1 - \tilde{v}_2\|\right)\left(\tilde{z}, m\right)$ is a function defined by $\tilde{v}_2(\tilde{z}, m) + \|\tilde{v}_1 - \tilde{v}_2\|$. Thus, for all $(\tilde{z}, m) \in \tilde{\mathbb{Z}} \times \mathbb{M}$:

$$\begin{aligned}
&\left(T\tilde{v}_1\right)\left(\tilde{z}, m\right) \\
&\quad \leq \sup_{\tilde{z}' \in \tilde{\Gamma}(\tilde{z}, m)} \left[\tilde{u}\left(\tilde{z}, \tilde{z}', m\right) + \beta \int_{\underline{m}}^{\overline{m}} \left(\tilde{v}_2\left(\tilde{z}', m'\right) + \|\tilde{v}_1 - \tilde{v}_2\|\right) dF\left(m'\right)\right] \\
&\quad = \sup_{\tilde{z}' \in \tilde{\Gamma}(\tilde{z}, m)} \left[\tilde{u}\left(\tilde{z}, \tilde{z}', m\right) + \beta \int_{\underline{m}}^{\overline{m}} \tilde{v}_2\left(\tilde{z}', m'\right) dF\left(m'\right)\right] + \beta \|\tilde{v}_1 - \tilde{v}_2\| \\
&\quad = \left(T\tilde{v}_2\right)\left(\tilde{z}, m\right) + \beta \|\tilde{v}_1 - \tilde{v}_2\| \qquad (44)
\end{aligned}$$

And similarly:

$$\left(T\tilde{v}_2\right)\left(\tilde{z}, m\right) \leq \left(T\tilde{v}_1\right)\left(\tilde{z}, m\right) + \beta \|\tilde{v}_1 - \tilde{v}_2\| \qquad (45)$$

for all $(\tilde{z}, m) \in \tilde{\mathbb{Z}} \times \mathbb{M}$. Now, (44) and (45) imply that for any $\tilde{v}_1, \tilde{v}_2 \in B(\tilde{\mathbb{X}})$:

$$\|T\tilde{v}_2 - T\tilde{v}_1\| \leq \beta \|\tilde{v}_1 - \tilde{v}_2\|$$

Thus, T is a contraction mapping with modulus β, taking $B(\tilde{\mathbb{X}})$ into itself.

Now, since $B(\tilde{\mathbb{X}})$ is a complete metric space, it again follows from the Contraction Mapping Theorem that T has a unique fixed point, $\tilde{v}' \in B(\tilde{\mathbb{X}})$. Furthermore, for any $\tilde{v}_0 \in B(\tilde{\mathbb{X}})$:

$$\|T^n \tilde{v}_0 - \tilde{v}'\| \leq \beta^n \|\tilde{v}_0 - \tilde{v}'\| \quad , \quad n = 0, 1, 2, ...$$

It remains to be shown that $\tilde{v}'$ corresponds to the supremum function, $\tilde{v}^*$, from the corresponding sequence problem. Since this proof requires a lot of extra notation, without providing much additional insight, the reader is referred to Chapter 9 in Stokey and Lucas (1989), where a more general proof of this is provided. ■

Proof of Proposition 8. First, it is argued that the mapping T takes the space of bounded functions on $\tilde{\mathbb{X}} = \tilde{\mathbb{Z}}_b \times \mathbb{M}$ into itself. Secondly, this result is extended to the space of bounded and continuous functions. Finally, the Contraction Mapping Theorem is applied to show the remaining results of the Proposition.

The mapping T was defined by:

$$(T\tilde{v})(\tilde{z}, m) = \sup_{\tilde{z}' \in \tilde{\Gamma}(\tilde{z}, m)} \left[\tilde{u}(\tilde{z}, \tilde{z}', m) + \beta \int_{\underline{m}}^{\overline{m}} \tilde{v}(\tilde{z}', m') \, dF(m') \right] \tag{46}$$

where $\tilde{\Gamma}(\tilde{z}, m)$ is non-empty and compact for any $\tilde{z} \in \tilde{\mathbb{Z}}_b$. If $\tilde{s}^{min} > 0$ and $r > \alpha$, then $\tilde{c}' = (1 + \alpha)^{-1} (r - \alpha) \tilde{s}^{min} > 0$ is a feasible strategy for all $\tilde{\Theta} \geq 0$ and $\tilde{s} \geq \tilde{s}^{min}$. Alternatively, if $\tilde{\Theta}^{min} > 0$, then $\tilde{c}' = (1 + \alpha)^{-1} \underline{m} \tilde{\Theta}^{min} > 0$ is feasible for all $\tilde{\Theta} \geq \tilde{\Theta}^{min}$ and $\tilde{s} \geq 0$. Since it is also bounded from above, then, for any $\tilde{v} \in B(\tilde{\mathbb{X}})$, the mapping T in (46) yields another bounded function, $T : B(\tilde{\mathbb{X}}) \rightarrow B(\tilde{\mathbb{X}})$. Choosing $\tilde{c} = 0$ can therefore never be optimal.

To show that $(T\tilde{v})(\tilde{z}, m)$ is also continuous if $\tilde{v}(\tilde{z}, m) \in C(\tilde{\mathbb{X}})$, one can proceed as in the proof of Proposition 3 by noting that $\int_{\underline{m}}^{\overline{m}} \tilde{v}(\tilde{z}', m') \, dF(m')$ is independent of m and continuous in $\tilde{z}'$ when $\tilde{v}(\tilde{z}', m') \in C(\tilde{\mathbb{X}})$.

This latter property is shown by considering a sequence $\tilde{z}'_n \rightarrow \tilde{z}'$. It follows, that

$$\left| \int_{\underline{m}}^{\overline{m}} \tilde{v}(\tilde{z}'_n, m') \, dF(m') - \int_{\underline{m}}^{\overline{m}} \tilde{v}(\tilde{z}', m') \, dF(m') \right| \leq \int_{\underline{m}}^{\overline{m}} |\tilde{v}(\tilde{z}'_n, m') - \tilde{v}(\tilde{z}', m')| \, dF(m')$$

And since $\tilde{z}'_n \to \tilde{z}'$, a compact set $\bar{\mathbb{X}} = \mathbb{D} \times \mathbb{M}$ can be constructed such that $\tilde{z}'_n \in \mathbb{D}$, $\tilde{z}' \in \mathbb{D}$, and $\mathbb{D} \subseteq \tilde{\mathbb{Z}}_b$. Now, since $\tilde{v}(\tilde{z}'_n, m')$ is continuous on $\bar{\mathbb{X}}$, it must also be uniformly continuous on $\bar{\mathbb{X}}$, since $\bar{\mathbb{X}}$ is compact. This implies that for every $\varepsilon > 0$, there exists an N_ε such that:

$$|\tilde{v}(\tilde{z}'_n, m') - \tilde{v}(\tilde{z}', m')| < \varepsilon$$

for all $n \geq N_\varepsilon$ and all $m' \in \mathbb{M}$. Thus, $\int_{\underline{m}}^{\overline{m}} \tilde{v}(\tilde{z}', m') \, dF(m')$ must be continuous in $\tilde{z}'$ when $\tilde{v}(\tilde{z}', m') \in C(\tilde{\mathbb{X}})$.

Finally, it follows from the proof of Proposition 7 that T is a contraction mapping with modulus β. Thus, the Contraction Mapping Theorem can be applied to complete the proof. ■

Proof of Proposition 9.

This proof is completely analogous to the proof of Proposition 5. The only change is that the deterministic value functions are now replaced by expected value functions. ■

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