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Quasi-static Macroeconomic Systems

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Abstract

This paper applies quasi-static analysis to a simple closed macroeconomy. It is shown that if the economy satisfies a conservation of income requirement, and the requirement that all equivalent investments generate the same rate of return (non-arbitrage), then there exists a state variable which measures the opportunity cost of moving from one macroeconomic equilibrium to another. This state variable is an *economic constraint* which measures the expenditure necessary to change equilibria. Central to this analysis is a definition of *economic time*, which is an invariant quantity with respect to the state variables used as a frame of reference.

1 Introduction

This is an application of the theory of quasi-static systems to macroeconomic analysis. Quasi-static systems are defined as those which change state while always remaining in equilibrium. If taken literally they most closely approximate the Classical and Neo-classical schools of thought on macroeconomic dynamics, where prices and quantities adjust instantaneously and frictionlessly to equilibrium in the absence of "shocks". Disequilibrium analysis, where an economy may persist for many periods without market clearing, is excluded by assumption.

This paper is decidedly not, however, an appeal to adopt the (Neo)Classical approach to macroeconomic analysis: in fact, insofar as quasi-static systems have an importance in and of themselves it is simply that they are the framework within which to build a methodology of examining the movements of aggregate variables (or average variables, if the number of individual components of the system be

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known). Quasi-static systems are used in this context to underscore a feature of the macroeconomy which is often ignored in modern macroeconomic modeling: the macroeconomy is a collection of a huge number of economic participants engaging in an enormous number of trades, all of which cannot possibly be identified or catalogued in practice. While it is clear that some abstraction is necessary from a modeling perspective, macroeconomic modeling usually swings far in the other direction. Current models tend to place such restrictions upon the actual behavior of the economic participants that the underlying diversity of the microeconomy becomes irrelevant. Agents are assumed to behave so that representative agent theory holds, the "correct" equilibrium is a steady-state with purely exogenous shocks, and so on. By assuming away the diversity of the underlying microeconomy, the resulting macroeconomy is often bereft of either interesting dynamics or "real-world" behavior.

The application of quasi-static systems outlined here is, in the author's opinion, a stepping-stone away from this trend and towards a more realistic combination of the incredible diversity of the microeconomy with their macroeconomic averages. It is a stepping-stone because it is an introduction to the tools and techniques of quasi-static analysis, including non-exact differentials over a business cycle, the existence of a measurement of "constraint" in an economic system (which turns out to be a form of opportunity cost), and a simple characterization of the real interest rate as the primary vehicle for non-arbitrage. Underlying all the results in the paper is a microeconomic interpretation which uses the diversity of the microeconomy to its fullest advantage–put simply, the interpretation uses the power of the Average in describing movements in macroeconomic variables. This interpretation is discussed more fully in a companion paper.

The simple model outlined below is built upon two assumptions about any macroeconomy:

1. Spending during a quasi-static change from one equilibrium to another may come either from savings or from income,
2. The law of non-arbitrage holds: funds will flow between portfolio holdings until their rates of return are equal.

Under these assumptions we may state the following: any equilibrium state of the economy may be characterized by the opportunity cost of arriving at that equilibrium from another. The magnitude of this opportunity cost defines how "constrained" the economy is relative to the initial equilibrium. Moreover, this *economic constraint* is a state variable describing the equilibrium, so that comparative statics relating e.g. consumption, income and the price level may be derived.

In order to arrive at this characterization, a definition of *economic time* is introduced which links the real rate of interest with a (normalized) path length of an equilibrium transition. It is shown that defining an interval of time in this fashion implies that for smaller and smaller real interest rates and a fixed value of consumption, either high prices are associated with low consumption (a high-inflation regime) or low prices are associated with high consumption (a low-inflation regime). In addition, there is a single equilibrium for which the real interest rate is maximized, where the high-inflation and low-inflation regimes coincide.

The structure of the paper is as follows: Section 2 introduces quasi-static systems analysis for a very simple economy, defines a business cycle, and states the law of the conservation of income. Section 3 states the law of non-arbitrage for this economy, and introduces the economic constraint. Section 4 defines economic time and identifies the real interest rate as a state variable. Section 5 formally derives the economic constraint, while Section 6 places the economic constraint within the context of equilibrium opportunity cost, and derives the income equation. Section 7 concludes.

2 Quasi-static Systems and the Business Cycle

Equilibrium is dictated by conservation (or preservation) of some quality or qualities over time. In the case of an economic system, it can be difficult to define just what is conserved. For example, it is often a difficult problem to separate variables which are actually stationary (or 'steady state') from those which are slowly-moving but ever-changing. Qualities of an economic system which are generally considered non-stationary include productivity and per capita income, both of which may be used to define economic *growth*. Nonetheless, it can be assumed that certain characteristics of an economic system are slowly-varying enough that, over the short run, they may be considered as stationary quantities. Such abstraction is of course quite natural in economic theory, where great pains are taken to separate the true (assumed stationary) links of causation between variables from the (assumed exogenous) 'background noise' of the economy. Indeed, one often finds examples in the literature where non-stationary time series is 'detrended' in order to make the resulting residuals stationary.

This paper investigates the implications of the assumption that in a closed economy, aggregate income may be a conserved quantity. There is no increase in productivity to introduce a trend in income growth–nor is there an international transmission mechanism of any kind to augment or diminish aggregate income for the domestic economy. This is an economy the way it is taught in introductory Macroeconomics: a fixed level of income which is divided into e.g. consumption, savings, taxes, etc. and which in the long run may adopt its Classically fixed value determined by the underlying technology of the economy. If income is conserved, then the entire economic system is itself a conservative system which possesses some global structure (inherited from the underlying microeconomic foundations). It is the goal of the paper to introduce a methodology by which this structure may be more thoroughly analyzed, while at the same time providing a measure of how easy or difficult an economy may move from one state to another. This measure is in a sense universal to any conservative economic system.

2.1 Income as a Conserved Quantity

We take as our starting point the idea that an economy may be represented by a certain collection of *state variables*, and that these variables together define the state of the economy at any point in time. To keep matters as simple as possible, we shall assume that the variables which most often (but not exclusively) define an economy's state are 1) the real level of aggregate income y, 2) the real level of aggregate consumption c, and 3) the price level p. It is clear that this is a non-exhaustive list since (for example) aggregate savings s may be defined as a state variable from the relation

$$s = y - c, \tag{2.1}$$

in a simple closed economy without government. But what is meant by state variable is that given, say, a savings level s and a consumption level c, there exists one (and only one) income level y which is defined by s and c. In other words, real income is a function of state:

$$y = y(s, c). \tag{2.2}$$

Notice that this is simply equation (2.1) where $y(s, c) := s + c$. For the simple system which we are investigating, all functions of state are expressible as functions of two state variables (more complicated systems may, naturally, have functions of state which depend upon more than two state variables–we are examining the simplest case for the sake of exposition).

Having defined the state variables, we wish to analyze how they change, i.e. what they depend upon. In order to do so it is necessary to define what types of changes will be admissible. This is the key feature of the quasi-static analysis: the economic system is assumed to be *slowly-varying*. To put this another way, any small amount of consumption or savings added to or removed from the system is performed in such a way as to preserve the (actually static) equilibrium at every point in time. This type of system, in which the dynamic treatment is considered to be slowly-varying with respect to its impact upon the state variables, is called a "quasi-static" system. It is as though any change in the features of an economy actually traces out a sequence of equilibrium states, starting from the equilibrium state before the change and arriving at some final equilibrium state. Indeed, we may represent the changes in a quasi-static system by paths in the state space. In Figure (1), for example, the path traced between equilibrium point A and equilibrium point B means that on its way from A to B, the economy may always be represented by equilibrium state variables along the way. The economy is never in disequilibrium.

Notice that in going from points A to B in Figure (1), the price level has dropped from p_A to p_B, while the consumption level has risen from c_A to c_B. The economy has "done something" (or has had something done to it) which has moved it from one equilibrium state to another. How has this been accomplished? We note that in moving from $(c_A, p_{A)}$ to (c_B, p_B) there must be some change in spending behavior–either less money is being spent on the new consumption bundle (in which case additional income has been saved), or more money has been spent (in which case additional income has been spent). In fact, this pattern is more general: the change in spending behavior will depend upon the path which is taken between points A and B. In other words, the amount spent (or saved) will depend not only upon the initial and final equilibrium states, but also upon the equilibrium states that the economy occupies in between.

This must be the case precisely because the system is assumed to be quasi-static–the economy cannot magically jump from point A to point B, with a simple adjustment to spending given by, say, $p_B c_B - p_A c_A$. Rather, the economy must progress through all the stages in between, resulting in adjustments to spending along every point in the path. What are these incremental adjustments along the way? They are simply the value of each incremental change in consumption, i.e. pdc for an incremental change in consumption dc. Thus, we may define this adjustment to spending (or more descriptively, the adjustment to expenditure) in the following way:

Definition 1 *The adjustment to expenditure δE from an initial equilibrium point A to a final equilibrium point B is given by*

$$\delta E := \int_{path(A,B)} pdc, \tag{2.3}$$

where the integral is taken over the path in the state space from A to B.

Note that according to this definition, a change in the equilibrium state which preserves the value of consumption, i.e. a change for which it is always true that $pc = V$ for some constant V, still implies an adjustment to expenditure δE which may be non-zero! This again reflects the fact that in order to move from one equilibrium to another it is not simply the value of consumption which matters–it is how the economy moves from one equilibrium to another that determines how "easy" or "difficult" such a move may be. In this case, an economy which moves along an equilibrium path of constant consumption value may perhaps be spending less than another economy which chooses a different path. But this is not the same thing as saying that the first economy spends nothing in moving from one equilibrium to another. Rather, all economies adjust spending when they change their state.

Where do these adjustments to expenditure come from, and where do they go? The answer to this question is actually an assumption about how the economy operates in a simple system, and it is the far-reaching implications of this assumption (and one other) that the remainder of this paper focuses upon. It is assumed that nominal income Y is a conserved quantity–it may either be spent as an adjustment to expenditure, or it may be saved, in which case it is an adjustment to savings. That is,

Axiom 1 (Conservation of Income) *Nominal income Y is a conserved quantity–for a change in nominal income dY,*

$$dY = \delta E + \delta S, \tag{2.4}$$

where δE is the adjustment to expenditure, and δS is the adjustment to savings, defined as the residual of δE from dY.

Why nominal income, and not real income? The reason is that adjustments to expenditure take place in nominal terms, and it is easier to analyze the system in nominal rather than real terms. However, we shall assume that real and nominal income are related by

$$Y = p_y y, \tag{2.5}$$

where p_y is a kind of producer (or output) price index. Furthermore, to ease exposition (and to lay to one side an interesting investigation into the correlation between the consumer price index and the producer price index) we shall assume that the producer price index is constant with respect to change in state:

$$dY = p_y dy, \tag{2.6}$$

so that we may switch back and forth between nominal and real income. (Indeed, none of the forthcoming analysis would be different if p_y were simply normalized to 1, but we refrain from doing so in order to emphasize the fact that a more general model must incorporate the value of output as well as consumption.)

2.2 A Business Cycle

The quasi-static approach we shall adopt in this paper borrows very heavily from the systematic analysis of aggregate conservative systems performed in the mid 19th-century, by Maxwell, Boltzmann, Clausius, Carnot and many others. These researchers were concerned with conservative physical systems in which the internal energy of the system is conserved. What is striking, and particularly useful from the perspective of economic theory, is that this physical system, and the quasi-static economic system outlined previously, may be analyzed using nearly identical techniques. This paper is one such application of these techniques and their implications for macroeconomic modeling. The first such application we shall investigate is a simple model of a business cycle.

In this simple exposition we shall assume that a business cycle is truly a complete cycle, that is, the economy ends up at the end of the cycle precisely where it began. Suppose that the economy begins at an equilibrium point (c_A, p_A), and proceeds as given by the arrows in Figure (2). First there is an upswing in the business cycle (an economic expansion) funded solely by drawing down savings (so that income is held constant). Prices and consumption both rise to (c_B, p_B). Next the value of consumption is held constant as prices fall but consumption continues to rise, to (c_C, p_C)–here the expansion is funded only by income growth, and the adjustment to savings is zero. This concludes the expansionary phase of the business cycle. From this point the level of savings is increased, holding the new income level constant, to reduce both consumption and the price level until the value of consumption equals its value at the beginning of the cycle–the equilibrium moves to (c_D, p_D). And lastly, both the value of consumption and savings are held constant while income is reduced. Prices rise and consumption falls until the original equilibrium (c_A, p_A) is restored.

Using our earlier definitions of adjustments to expenditure and savings we can define the total adjustment to expenditure which has occurred over this business cycle. This is the amount which has been spent in order to drive the economy through the cycle. Once again, the fact that the economy ends up where it began is not the same thing as saying that "nothing has happened" over the business cycle–and it is precisely the adjustment to expenditure which measures how easy or difficult it is for an economy to be driven through a cycle such as the one depicted.

As defined, the total adjustment to savings and the total change in income must be sufficient to cover the total adjustment to expenditure over the business cycle:

$$\delta S_{A\to B} + dY_{B\to C} + \delta S_{C\to D} + dY_{D\to A} = -\delta E_{cycle} = -\int_{cycle} pdc. \tag{2.7}$$

Since income is a function of state, it is unchanged over the business cycle–this leads to the result that

$$\delta S_{cycle} = -\int_{cycle} pdc. \tag{2.8}$$

The amount of expenditure necessary to drive the economy over the business cycle is thus equal to the adjustment to savings over the cycle. Savings in this sense can measure or reflect how high (or low) an adjustment to expenditure needs to be in order for the economy to move from one equilibrium state to another. In this simple example, we see from the figure that the adjustment to expenditure is simply the area enclosed by the cycle. Hence, we would expect that smaller cycles (lower peaks and troughs) would necessitate less expenditure adjustment. In other words, a smaller adjustment to savings would be needed. This indicates the path dependence of expenditure and savings.

However, this measurement of the adjustment to expenditure, and the concomitant adjustment to savings, can be generalized–in fact, we can identify for each equilibrium point how easy or difficult it is to move from that point (i.e., we can measure how great or how small the adjustment to expenditure or savings must be for a departure from equilibrium). This is an extremely valuable quantity because it associates with any economic equilibrium a measure of how much "effort" is needed to move the economy between equilibria. For example, such a measure might be used to evaluate just how difficult a departure from a state of chronically high poverty will be, in terms of resources which must be expended by the economy along the way. On the other hand, if only certain limited resources are available to effect a change in the economy, then this measure would enable one to describe where the "best" equilibrium is located (e.g., the equilibrium which has the highest income, or highest consumption, or lowest, price, etc., attainable from the resources available).

On a completely different level this quantity, which we shall interchangeably refer to as the *economic constraint* or the economy's *opportunity cost*, has a much fuller interpretation with respect to the underlying microeconomic activity. As detailed in a companion paper, the economic constraint is itself defined by how easy it is for an economy to adjust its underlying microeconomic states. Thus, while the economic constraint can outline how an economy might move from one state to another, it is the connection of the constraint to the microeconomy which tells us *how* such a move changes the underlying microeconomy, and it identifies *why* some changes might be more difficult than others. The key point here is that changes in the macroeconomic state cannot be separated from, and are indeed defined by, those changes in the microeconomic states which define it. And as might be expected from economies which possess such a multitude of diversity, these microeconomic states are best defined as averages of uncertain economic behavior.

Thus, the current exposition in this paper is really an introduction to a prelude. It introduces a methodology which is too simple for the problem at hand, but which is readily expanded to 1) encompass the great heterogeneity of the underlying microeconomy, and 2) describe changes in equilibrium which are not quasi-static, i.e. non-conservative, in a way which can be linked to the simple model. We shall develop this methodology further by finding a representation for the economic constraint–note that adjustments to savings or expenditures by themselves are unable to play this role, as they are path dependent. What we seek is a measurement of constraint which is independent of the path the economy takes–that is, we seek a function of state which nonetheless is able to measure how much must be spent (or saved) in order to move the economy from a give equilibrium point. In order to accomplish this goal we shall need an assumption about the nature of economic equilibrium, one which is pervasive in the economic and financial economic literature and which holds an esteemed position in the current theory.

3 The Law of Non-arbitrage

By the first axiom, the change in nominal income dY must be equal to the sum of the adjustment to savings δS and the adjustment to expenditure δE :

$$dY = \delta S + \delta E. \tag{3.1}$$

We may express real income y as a function of consumption c and the real interest rate r, in which case equation (3.1) may be expressed in differential form as

$$\delta S = p_y \left[\left(\frac{\partial y}{\partial c} \right)_r dc + \left(\frac{\partial y}{\partial r} \right)_c dr \right] - p dc, \tag{3.2}$$

where e.g. $\left(\frac{\partial y}{\partial c} \right)_r$ represents the partial derivative of y with respect to c, holding r constant.

Suppose we consider two economies which are isolated except for capital (i.e. savings) flows. We suppose that these economies have been linked for some time, and we would expect that the law of non-arbitrage would hold between them, i.e. their respective real interest rates must be equal. Note that we are assuming the quasi-static framework in this statement, because we are "waiting long enough" for any initial differences in the rates of return to even out. This is a standard assumption used when discussing perfect capital markets–if arbitrage opportunities existed, then it would be possible to play one investment against the other and earn something from nothing. In a world with perfect capital markets and no uncertainty, such opportunities ought not to exist. This is the motivation behind introducing the law of non-arbitrage as a second fundamental assumption:

Axiom 2 (The Law of Non-arbitrage) *The rates of return on any two investments denominated in the same unit of account are equal.*

Thus, looking at the two economies as one economic system, we see that the state of the collective system may be described by e.g. c_1, the consumption in the first economy, c_2, the consumption in the second economy, and $r = r_1 = r_2$, the (identical) real interest rate in each economy. Consider now an adjustment to savings in each country, δS_i, $i = 1, 2$. For the complete system it is clear that since these economies are otherwise isolated, the total adjustment to savings for the collective system must be

$$\delta S = \delta S_1 + \delta S_2. \tag{3.3}$$

By substituting equation (3.2) for each economy into (3.3) we see that

$$\delta S = \left[p_{y_1}\left(\frac{\partial y_1}{\partial c_1}\right)_r - p_1\right]dc_1 + p_{y_1}\left(\frac{\partial y_1}{\partial r}\right)_{c_1} dr + \left[p_{y_2}\left(\frac{\partial y_2}{\partial c_2}\right)_r - p_2\right]dc_2 + p_{y_2}\left(\frac{\partial y_2}{\partial r}\right)_{c_2} dr \tag{3.4}$$

or

$$\delta S = \left[p_{y_1}\left(\frac{\partial y_1}{\partial c_1}\right)_r - p_1\right]dc_1 + \left[p_{y_2}\left(\frac{\partial y_2}{\partial c_2}\right)_r - p_2\right]dc_2 + \left[p_{y_1}\left(\frac{\partial y_1}{\partial r}\right)_{c_1} + p_{y_2}\left(\frac{\partial y_2}{\partial r}\right)_{c_2}\right]dr. \tag{3.5}$$

We now have the adjustment to savings, which is an inexact differential, expressed as a function of the exact differentials dc_1, dc_2 and dr. In order to evaluate this expression we use Carathéodory's principle, a version of which is stated as follows.[1]

Theorem 1 (Carathéodory's principle) *If an inexact differential expressed as*

$$\delta A = A_1 dx_1 + A_2 dx_2 + \ldots + A_n dx_n \tag{3.6}$$

has the property that, in any neighborhood of a point $x = (x_1, x_2, \ldots x_n)$ *there exists another point* $x' = (x'_1, x'_2, \ldots, x'_n)$ *which is inaccessible along the solution trajectory of* $\delta A = 0$ *through* x*, then there exists an integrating factor* $g(x_1, x_2, \ldots, x_n)$ *such that*

$$g(x)dX = \delta A = A_1 dx_1 + A_2 dx_2 + \ldots + A_n dx_n, \tag{3.7}$$

where dX *is an exact differential of a function of state* $X(x_1, x_2, \ldots, x_n)$.

[1] This version of Carathéodory's Principle is taken from Reiss (1965), p.26, as is the following exposition for the derivation of the function of state X_S. It must also be said here that the interpretation of the opportunity cost (to be defined later) as an economic constraint owes a great debt to Reiss' interpretation of entropy as a 'degree of constraint' in quasi-static thermodynamic systems.

Carathéodory's principle tells us that if solutions to $\delta S = 0$ somehow restrict the allowable states of the system, then we may find an integrating factor $g(c_1, c_2, r)$ such that $gdX = \delta S$. We will then have found the function of state X which defines, for every equilibrium, a level of constraint felt by the economy at that equilibrium (as discussed briefly in Section 2 and in further detail in Section 6). Is Carathéodory's principle applicable to this economy? Consider again the prototypical business cycle in Figure (2), where the economy passes through two stages of isoincome growth and contraction, and two stages of isovalue growth and contraction. It is precisely along the isovalue paths that $\delta S = 0$, so that adjustments to expenditure are applied solely from changes in income dY. Clearly, any deviation off the isovalue path will by Axiom (1) require an adjustment to savings so that $\delta S \neq 0$. In other words it is not possible to reach any other point in the state space with $\delta S = 0$ while at the same time allowing pc to vary. Thus, there are points in any arbitrary neighborhood around $pc = V$, V a positive constant, which are inaccessible to solutions where δS remains equal to zero.

From this argument we may apply Carathéodory's principle and conclude that there must exist an integrating factor $g(c_1, c_2, r)$ such that

$$\delta S = g(c_1, c_2, r)dX, \tag{3.8}$$

for the composite system, and

$$\begin{aligned} \delta S_1 &= g_1(c_1, r)dX_1, & (3.9) \\ \delta S_2 &= g_2(c_2, r)dX_2. & (3.10) \end{aligned}$$

for each individual system. Note that each adjustment to savings is only dependent upon the states of that system, so that g_1 and g_2 only depend upon c_1 and r and c_2 and r, respectively.

Substitution of equation (3.8), (3.9) and (3.10) into (3.3) yields

$$g(c_1, c_2, r)dX = g_1(c_1, r)dX_1 + g_2(c_2, r)dX_2. \tag{3.11}$$

Since X, g, g_1 and g_2 are all functions of state they are expressible as $X(X_1, X_2, r)$, $g(X_1, X_2, r)$, $g_1(X_1, r)$ and $g_2(X_2, r)$, respectively. And since dX is an exact differential, it must be true that

$$\frac{\partial}{\partial r}\left(\frac{g_1(X_1, r)}{g(X_1, X_2, r)}\right)_{X_1, X_2} = \frac{\partial}{\partial X_1}(0)_{r, X_2} = 0, \tag{3.12}$$

$$\frac{\partial}{\partial r}\left(\frac{g_2(X_2, r)}{g(X_1, X_2, r)}\right)_{X_2, X_1} = \frac{\partial}{\partial X_2}(0)_{r, X_1} = 0. \tag{3.13}$$

Combining these expressions yields

$$\frac{\partial g_1(X_1, r)/\partial r}{g_1(X_1, r)} = \frac{\partial g_2(X_2, r)/\partial r}{g_2(X_2, r)} = \frac{\partial g(X_1, X_2, r)/\partial r}{g(X_1, X_2, r)}. \tag{3.14}$$

It is clear from the above expression that the functions $\frac{\partial g_1(X_1,r)/\partial r}{g_1(X_1,r)}$, $\frac{\partial g_2(X_2,r)/\partial r}{g_2(X_2,r)}$ and $\frac{\partial g(X_1,X_2,r)/\partial r}{g(X_1,X_2,r)}$ cannot, in fact, depend upon either X_1 or X_2–in fact we can say more because the only thing which has functionally distinguished g_1 and g_2 from g (and hence, from each other) has been this dependence upon X_1 or X_2.

What is the resulting expression for the adjustment to savings δS? We notice from relation (3.14) above that

$$\frac{\partial g/\partial r}{g} = \gamma(r) \Longrightarrow \tag{3.15}$$

$$\ln(g) = \int \gamma(r)dr + \Gamma(X) \Longrightarrow \tag{3.16}$$

$$g = e^{\int \gamma(r)dr} e^{\Gamma(X)}, \tag{3.17}$$

where $\Gamma(X)$ is an arbitrary function of state. Let $\bar{\Gamma} := e^{\Gamma(X)}$ and $f(r) := e^{\int \gamma(r)dr}$. Then we may express the integrating factor as

$$g(X,r) = f(r)\bar{\Gamma}. \tag{3.18}$$

In summary, then, we may express δS as

$$\delta S = f(r)\bar{\Gamma}dX. \tag{3.19}$$

If we now define the function of state which represents the degree of constraint for an economy as

$$X_S := \int \bar{\Gamma}dX, \tag{3.20}$$

then we may relate this function of state to the adjustment to savings by

$$\delta S = f(r)dX_S. \tag{3.21}$$

Our task is nearly complete–what we have shown so far is that there exists a function of state which at a given equilibrium measures, independent of path, the adjustment to expenditure (or adjustment to savings) which occurs following a change from one equilibrium to another. As this measurement is a function of state, it is intimately related to other functions of state such as income, consumption, the price level, etc. through various transformations which will be examined in Section 6. The only part of the goal left undone is to identify how the integrating factor $f(r)$ depends upon the real interest rate.

In order to do this let us reintroduce equation (3.2), which ties together changes in income with adjustments to savings and expenditure:

$$\delta S = p_y \left[\left(\frac{\partial y}{\partial c} \right)_r dc + \left(\frac{\partial y}{\partial r} \right)_c dr \right] - pdc. \tag{3.22}$$

We now know that $\delta S = f(r) dX_S$, so that

$$dX_S = \frac{\left[p_y \left(\frac{\partial y}{\partial c} \right)_r - p \right]}{f(r)} dc + \frac{p_y \left(\frac{\partial y}{\partial r} \right)_c}{f(r)} dr. \tag{3.23}$$

Once again we have an exact differential, so it must be true that

$$\frac{\partial}{\partial r} \left(\frac{\left[p_y \left(\frac{\partial y}{\partial c} \right)_r - p \right]}{f(r)} \right)_c = \frac{\partial}{\partial c} \left(\frac{p_y \left(\frac{\partial y}{\partial r} \right)_c}{f(r)} \right)_r \implies \tag{3.24}$$

$$-\frac{f'(r)}{f(r)^2} \left[p_y \left(\frac{\partial y}{\partial c} \right)_r - p \right] + \frac{1}{f(r)} \left[p_y \left(\frac{\partial^2 y}{\partial c \partial r} \right)_{r,c} - \left(\frac{\partial p}{\partial r} \right)_c \right] = \frac{1}{f(r)} p_y \left(\frac{\partial^2 y}{\partial r \partial c} \right)_{c,r} \tag{3.25}$$

or (since dy is an exact differential)

$$-f'(r) \left[p_y \left(\frac{\partial y}{\partial c} \right)_r - p \right] - f(r) \left(\frac{\partial p}{\partial r} \right)_c = 0. \tag{3.26}$$

In order to evaluate this expression there must be a relationship between the real interest rate r, and the state variables y, c, and p. Defining this relationship highlights the nature of *economic time* in this system, so we shall devote a separate section to it en route to the final specification of the integrating factor.

4 Economic Time and the Interest Rate

The role of the interest rate in an economy is simply to represent the possible opportunities forgone when one investment is made at the expense of another. In other words, interest is the opportunity cost of investment measured as an incremental (or instantaneous) return *over time.* Thus what is needed is a relationship between the time state of the system, which is not modeled explicitly, and the economic state of the system. The key assumption here is that *longer paths in state space take a longer amount of time to traverse.* Under this assumption we may link time changes with changes in state variables and hence the level of the interest rate with changes in state variables.

Consider the small change of state from point A to point B in Figure (3). Along this path we may define the real interest rate to be

$$r dt_{path(A,B)} = \frac{dc}{c}, \tag{4.1}$$

where $dt_{path(A,B)}$ is the interval of time taken to traverse the path between A and B, i.e. the amount of time to accumulate a real return dc, starting from c. The real interest rate is defined as an instantaneous real return. Thus, the incremental rate of return earned from holdings of real consumption over a small time interval dt, i.e. $\frac{dc}{c}$, must be equal to this instantaneous rate of return r over dt.

The amount of time taken to move from point A to point B, and hence the amount of time necessary for holdings of consumption to appreciate by the amount dc, depends upon the *path* between A and B. We might just as easily have defined another path from A to B as in Figure (4). Here, it takes much longer to arrive at the final appreciation amount dc due to the intermediate values the consumption good takes. This implies that the instantaneous interest rate over this time period will be less–in the limiting case where the path length between an initial equilibrium A and a final equilibrium B is infinite, we should expect the real interest rate to be zero: it takes an infinitely long time to accumulate an infinitesimal amount of the consumption good, which is no rate of return at all.

By now there may be the very reasonable objection from the reader that the interval of time just defined cannot have a unique value. For if one were to choose a different state space representation,using say income and consumption instead of price and consumption in Figure (3), then the resulting time interval may be longer or shorter than the one initially defined! In fact this is quite correct–the time interval must be normalized with respect to one path, in order to define it as an invariant quantity over all paths.

4.1 A side-note on the nominal interest rate

Before moving further in this direction it is useful to define the nominal interest rate along the way. We define the nominal interest rate i simply as the proportional change in the value of a particular holding of the consumption good, i.e. a change in pc :

$$\frac{d(pc)}{pc} = idt_{path}. \tag{4.2}$$

This corresponds with the intuitive notion that the nominal interest rate is dependent upon changes in the price level as well as changes in consumption. Indeed, performing the total differentiation of the left hand side we find

$$\frac{pdc + cdp}{pc} = idt_{path} \Longrightarrow \tag{4.3}$$

$$\frac{dc}{c} + \frac{dp}{p} = idt_{path}, \tag{4.4}$$

or

$$r + \pi = i, \tag{4.5}$$

where we have used the relation $rdt_{path} = \frac{dc}{c}$ and have defined a quantity $\pi dt_{path} = \frac{dp}{p}$, where π is the instantaneous rate of inflation. Thus we arrive at the usual definition of the real interest rate as the difference between the nominal interest rate and the rate of inflation:

$$r = i - \pi. \tag{4.6}$$

4.1.1 Normalization coefficients of economic time

Next consider an arbitrary path between two equilibrium states A and B. The arc-length s between A and B is defined as the length of the path between these two points. For an infinitesimal movement (dc, dp) along this path we may define a corresponding infinitesimal change in the arc-length as

$$(ds_{c,p})^2 = (dc)^2 + (dp)^2. \tag{4.7}$$

The economic time assumption stated earlier may then be represented as

$$dt_{path} \propto ds_{c,p}, \tag{4.8}$$

so that longer paths correspond to longer times spent between states A and B.

Now we need to address the problem of scaling. Suppose instead that we selected as our state variables (c, y) instead of (c, p). Then the arc-length under consideration would be

$$(ds_{c,y})^2 = (dc)^2 + (dy)^2. \tag{4.9}$$

The principle of invariance of economic time simply states that regardless of the state variables used to represent the economy (i.e., regardless of the *frame of reference* of state variables used), the measurement of economic time must be the same for the same economic change. That is,

$$dt_{path} = k_1 ds_{c,p} = k_2 ds_{c,y}, \tag{4.10}$$

where the k_i are normalizing coefficients. These coefficients are related by

$$\left(\frac{k_2}{k_1}\right)^2 = \left(\frac{ds_{c,p}}{ds_{c,y}}\right)^2 = \frac{(dc)^2 + (dp)^2}{(dc)^2 + (dy)^2} = \frac{(dc)^2\left(1 + \left(\frac{dp}{dc}\right)^2\right)}{(dc)^2\left(1 + \left(\frac{dy}{dc}\right)^2\right)} \Longrightarrow \tag{4.11}$$

$$\frac{k_2}{k_1} = \frac{1 + \left(\frac{dp}{dc}\right)^2}{1 + \left(\frac{dy}{dp}\right)^2\left(\frac{dp}{dc}\right)^2}. \tag{4.12}$$

One may express all representations arc-length of the same economic change (i.e., the same change dc) as ratios governed by the normalizing coefficients. As one might expect, these coefficients are determined only relative to each other–once a scale has been fixed, the rest are determined by the rules of transformation from one coordinate system to another. In this context, then, we are free to choose one coordinate system for which to fix our normalizing coefficient, and proceed from there. Note that although it is tempting to select, say, $k_1 = 1$ and define all other coefficients in this relative way, this approach will not identify the correct scaling quantity to be used. The value for any particular normalizing coefficient must (in a testable context) be determined by the data (discussed in the concluding remarks). The rest of the normalizing coefficients may then be found from such relations as (4.12).

Thus, we shall pick one coordinate system and stick with it–we shall select the (c, p) system and define the time interval over a particular path in that system as

$$dt_{path} = k_{c,p} ds, \tag{4.13}$$

where $k_{c,p}$ is the normalizing coefficient in the (c, p) system.

By substitution of (4.7) and (4.13) into(4.1) we arrive at

$$\frac{dc}{c} = k_{c,p} r \sqrt{(dc)^2 + (dp)^2}, \tag{4.14}$$

where the positive root of the arc-length is chosen for obvious reasons. Rearrangement of this relationship yields

$$\left(\frac{1}{c^2 (k_{c,p} r)^2} - 1 \right) (dc)^2 = (dp)^2 \Longrightarrow \tag{4.15}$$

$$\left(\frac{1}{c^2 (k_{c,p} r)^2} - 1 \right)^{1/2} dc = dp. \tag{4.16}$$

Thus we see that, as expected, the real interest rate is a function of state: if we specify, for example, the interest rate and the price level, then we have determined the level of consumption.

As an example let us consider what happens to the real and nominal interest rates along curves in the state space of constant value of consumption, i.e. curves for which

$$pc = V \tag{4.17}$$

for some constant $V > 0$.

What happens to the nominal interest rate in this case? By construction, if the economy moves along an isovalue line, so that equality (4.17) is preserved at all times, then there is no appreciation in the value of consumption holdings–hence, the nominal interest rate is zero. We may also observe this just by taking the total differential of (4.17):

$$\frac{dc}{c} + \frac{dp}{p} = 0 = i dt_{pc=V}, \tag{4.18}$$

from which

$$r dt_{pc=V} = \frac{dc}{c} = -\frac{dp}{p} = -\pi dt_{pc=V}. \tag{4.19}$$

What is the length of time that it takes to move along an isovalue curve an infinitesimal length (dc, dp)? From above it must be equal to the infinitesimal arc-length, which is computed from above as

$$(ds)^2 = (dc)^2 + \left(-\frac{p}{c}dc\right)^2 \tag{4.20}$$

$$= \left(1 + \frac{V^2}{c^4}\right)(dc)^2 \tag{4.21}$$

where we have first used (4.19) to express dp as a function of dc, and then used the isovalue relation $pc = V$.

The amount of time spent along a small section of the path along the isovalue curve is then

$$dt_{pc=V} = k_{c,p}ds = k_{c,p}dc\sqrt{1 + \frac{V^2}{c^4}} \tag{4.22}$$

from which we arrive at the relation[2]

$$k_{c,p}r\sqrt{1 + \frac{V^2}{c^4}} = \frac{1}{c}. \tag{4.23}$$

Before continuing to the solution (which simply involves substituting $pc = V$ back into the equation and solving for c) we can stop here to investigate the way in which, for a given value V, the consumption level depends upon the interest rate. This equation is a fourth-order polynomial in c, and thus has four roots. Two of these roots are negative, which we discard–the remaining two positive real roots are

$$c_1 = \left(\frac{1 - \sqrt{1 - 4V^2(k_{c,p}r)^2}}{2(k_{c,p}r)^2}\right)^{1/2}, \tag{4.24}$$

$$c_2 = \left(\frac{1 + \sqrt{1 - 4V^2(k_{c,p}r)^2}}{2(k_{c,p}r)^2}\right)^{1/2}. \tag{4.25}$$

These roots exist for all positive values of V and r. Notice that for each isovalue curve and each level of the real interest rate there exist two values of consumption c for which

$$r\,dt_{pc=V} = \frac{dc}{c}. \tag{4.26}$$

In the two-dimensional state space (c, p) it may be readily seen that these two points have an axis of symmetry around the line defined by the origin and the single point where $c_1 = c_2$, which is found to be

[2] Note that we are being a bit sloppy here: to be completely precise we should not simply cancel the differentials but should integrate both sides along the path dc from a reference point c^* to an arbitrary consumption level c and then compare the results. For small changes dc, however, the two approaches can be shown to yield identical results in this case, and we adopt the less precise but more compact method for the exposition.

$$c^* = \frac{\sqrt{2}}{2k_{c,p}r^*}. \tag{4.27}$$

The key point here is to note simply that for the isovalue lines, those values of consumption and price for which $\left(\frac{dp}{dc}\right)_{c=c_1} = -\left(\frac{dp}{dc}\right)_{c=c_2}$ must have the same interest rate! Intuitively it is clear that this must be the case, as the relative appreciation of the consumption good, when accounting for price movements, will be the same if the relative trade-off between consumption and the price level is the same. The second point to note is that as the value pc rises, i.e. as V increases, the interest rate at a given price level must fall. This is due to the fact that an infinitesimal change dc will become less and less significant, the larger is c. These two facts are simply reflected in the fact that for a given interest rate, the points (c, p) must lie on a circle, as in Figure (5). As r decreases, the points of intersection of the isovalue line with the constant-interest-rate circle must diverge further and further away from ray between the origin and c^*. This also implies that as the overall value of consumption increases, the maximum interest rate which can sustain this value (i.e. r^*) must fall. The lowest interest rate levels will exist at the "endpoints" of the isovalue curve, i.e. at those points where either 1) consumption is low and the price level is high (by the same argument, this is a low-inflationary state), or 2) consumption is high and the price level is low (which is a high-inflationary state). If we observe very low interest rate levels coupled with a high value of consumption V, then there are only two economic 'regimes' which can exist, one with high inflation and one with low inflation. Economies with a very low value of consumption, on the other hand, may exist at all interest rate levels.

Let us continue with the solution for the isovalue case: using the relation $pc = V$ we may resubstitute this back into (4.23) to yield

$$k_{c,p}r\sqrt{1 + \frac{p^2}{c^2}} = \frac{1}{c} \implies \tag{4.28}$$

$$c = \left(\frac{1}{(k_{c,p}r)^2} - p^2\right)^{1/2} \tag{4.29}$$

or

$$c^2 + p^2 = \frac{1}{(k_{c,p}r)^2}. \tag{4.30}$$

As stated before, the equilibrium values of c and p for which the interest rate is constant lie on the quarter-circle where c and p are both positive.

For our grand purpose of finding the unknown integrating factor what is important is the function of state for c–or equivalently for p, i.e.

$$p = \left(\frac{1}{(k_{c,p}r)^2} - c^2\right)^{1/2}, \tag{4.31}$$

which is simply rewriting equation (4.29).

5 The Economic Constraint

Recall that considering an arbitrary equilibrium path along which $\delta S = f(r)dX_S$ we arrived at the relation (3.26):

$$-f'(r)\left[p_y\left(\frac{\partial y}{\partial c}\right)_r - p\right] - f(r)\left(\frac{\partial p}{\partial r}\right)_c = 0. \tag{5.1}$$

This may be rewritten as

$$-\frac{f'(r)}{f(r)} = \frac{\left(\frac{\partial p}{\partial r}\right)_c}{p_y\left(\frac{\partial y}{\partial c}\right)_r - p}, \tag{5.2}$$

from which we see that since the left hand side depends only upon r, the right hand side must satisfy

$$\frac{\partial}{\partial c}\left(\frac{\left(\frac{\partial p}{\partial r}\right)_c}{p_y\left(\frac{\partial y}{\partial c}\right)_r - p}\right) = 0 \Longrightarrow \tag{5.3}$$

$$\frac{\partial^2 p}{\partial r \partial c}\left(p_y\left(\frac{\partial y}{\partial c}\right)_r - p\right) = \left(\frac{\partial p}{\partial r}\right)_c\left(p_y\left(\frac{\partial^2 y}{\partial c^2}\right) - \left(\frac{\partial p}{\partial c}\right)_r\right) \Longrightarrow \tag{5.4}$$

$$\frac{\frac{\partial^2 p}{\partial r \partial c}}{\left(\frac{\partial p}{\partial r}\right)_c} = \frac{\left(p_y\left(\frac{\partial^2 y}{\partial c^2}\right) - \left(\frac{\partial p}{\partial c}\right)_r\right)}{\left(p_y\left(\frac{\partial y}{\partial c}\right)_r - p\right)}. \tag{5.5}$$

This relation must be true along any path in the state space–in particular, it must also be true along the curve given by (4.30), i.e. along curves of constant real interest rate. This allows us to find several of the quantities present in equation (5.5), from $p = \left(\frac{1}{(k_{c,p}r)^2} - c^2\right)^{1/2}$:

$$\begin{aligned}
\left(\frac{\partial p}{\partial r}\right)_c &= -\left(\frac{1}{(k_{c,p}r)^2} - c^2\right)^{-1/2}\frac{1}{(k_{c,p})^2 r^3} = -\frac{1}{(k_{c,p})^2 p r^3}, &(5.6)\\
\left(\frac{\partial p}{\partial c}\right)_r &= -\left(\frac{1}{(k_{c,p}r)^2} - c^2\right)^{-1/2}(2c) = -\frac{c}{p}, &(5.7)\\
\frac{\partial^2 p}{\partial r \partial c} &= -\left(\frac{1}{(k_{c,p}r)^2} - c^2\right)^{-3/2}\frac{c}{(k_{c,p})^2 r^3} = -\frac{c}{(k_{c,p})^2 p^3 r^3}. &(5.8)
\end{aligned}$$

Mass substitution into (5.5) yields

$$\frac{c}{p^2} = \frac{p_y\left(\frac{\partial^2 y}{\partial c^2}\right) + \frac{c}{p}}{\left(p_y\left(\frac{\partial y}{\partial c}\right)_r - p\right)} \Longrightarrow \tag{5.9}$$

$$p_y \left(\frac{\partial^2 y}{\partial c^2}\right) + \frac{c}{p} = \frac{c}{p^2} p_y \left(\frac{\partial y}{\partial c}\right)_r - \frac{c}{p} \Longrightarrow \tag{5.10}$$

$$\left(\frac{\partial^2 y}{\partial c^2}\right) = \frac{c}{p^2} \left(\frac{\partial y}{\partial c}\right)_r - \frac{2c}{p_y p}. \tag{5.11}$$

Letting $h(c,r) := \left(\frac{\partial y}{\partial c}\right)_r$, this defines a first-order partial differential equation:

$$\frac{\partial h}{\partial c} = \frac{c}{p^2} h - \frac{2c}{p_y p}. \tag{5.12}$$

Solutions to this partial differential define a *family* of solutions which depend upon an arbitrary function of the real interest rate–in this case, the solution can be written as

$$h(c,r) = -\frac{c^2}{p_y p} - \mathbf{i} \frac{\Lambda(r)}{k_{c,p} r p}, \tag{5.13}$$

where Λ is the arbitrary function of r, and $\mathbf{i}$ is the imaginary number $\sqrt{-1}$.

For simplicity we shall consider only the solution for which $\Lambda(r) = 0$.[3] Under this simplification we find that

$$\left(\frac{\partial y}{\partial c}\right)_r = -\frac{c^2}{p_y p}. \tag{5.14}$$

We may now substitute this result, and the earlier result for $\left(\frac{\partial p}{\partial r}\right)_c$, into equation (5.2) to arrive at a solution for the integrating factor:

$$-\frac{f'(r)}{f(r)} = \frac{\frac{1}{(k_{c,p})^2 p r^3}}{\frac{c^2}{p} + p} = \frac{\frac{1}{(k_{c,p})^2 r^3}}{c^2 + p^2} = \frac{1}{r}. \tag{5.15}$$

Integration of both sides then yields

$$\ln(f(r)) = -\ln(r), \text{ or} \tag{5.16}$$

$$f(r) = \frac{1}{r}. \tag{5.17}$$

Our representation of the adjustment to savings is complete: since we know that $f(r)$ has the same functional form regardless of state, we have that

[3] Note that this is *not* the same thing as considering only real solutions–for it is perfectly reasonable that Λ too may depend upon $\mathbf{i}$ in such a way as to make the second term on the right hand side real as well.

$$\delta S = \frac{1}{r} dX_S. \tag{5.18}$$

This is the fundamental equation we seek–it relates the adjustment to savings (and hence, as described earlier, the adjustments to expenditure) which occurs during a change of state to the state of the economy which exists just prior to the change. Measurement of the path dependent adjustment is now possible by measuring a change of state, which is path independent. Some implications of this result are outlined in the next section–they are not exhaustive by any means.

6 Economic Implications

The relationship (5.18) between the economic constraint X_S and the adjustment to savings is in fact a definition of *opportunity cost*. Consider a move from one equilibrium, say α, to another equilibrium β. The change in economic constraint may be given as

$$X_s(\beta) - X_S(\alpha) = \int_\alpha^\beta r\delta S. \tag{6.1}$$

Thus, the change in the economic constraint measures the return of the adjustment to savings between states α and β, and this change is independent of the path taken between α and β. Suppose that, for a fixed level of income, the movement from α to β required a positive adjustment to expenditure (so that $\delta E > 0$). For the same movement, then, the adjustment to savings was negative (since in this case $\delta E + \delta S = 0$), and the economic constraint falls from α to β. This fall may be *interpreted* as the return foregone by adjusting δE upwards instead of investing the same amount at the real interest rate r. Note that this remains in the realm of interpretation because we have said nothing about how savings is invested, or a return accrued, except with regard to the non-arbitrage law and the definition of real consumption accumulation. The return foregone is the opportunity cost of moving from equilibrium state α to equilibrium state β.

If the change in the economic constraint is negative, then, there is a cost to moving from one equilibrium to another, while if the constraint is positive there is an excess expenditure which is absorbed into savings and generates a return. It is intuitively appealing to refer to those equilibria which induce an opportunity cost (rather than a return) from a given reference state as more constrained precisely because they require an application of expenditure to generate. Somehow, something must be spent or sacrificed in order to obtain the new equilibrium–in other words, more conditions are imposed upon the system which require additional adjustments to expenditure.

Note that with this interpretation we must be very careful to distinguish between adjustments to expenditure which generate a net investment (generating a real opportunity cost), and those adjustments which would not lead to a net investment *because the interest rate changes at the same time*. For example, since the economic constraint is a function of state, over any quasi-static business cycle the net change in this constraint is zero, even though adjustments to expenditure are made along the way. The net change in the opportunity cost over any quasi-static business cycle is zero, because the interest rate adjusts so as to balance any gains or losses along the way:

$$\Delta X_s = \int_{cycle} r\delta S = 0. \tag{6.2}$$

Thus we can interpret the economic constraint X_S as an opportunity cost of moving from one equilibrium state to another equilibrium state. While it is certainly plausible to infer that at a given equilibrium state this opportunity cost is minimized (so that any infinitesimal deviation of X_S away from its equilibrium value is zero), this does not appear to be deducible from the foregoing analysis. A deeper investigation into the microeconomic foundations of the opportunity cost are necessary before such a statement can be rigorously derived.

We are now in a position to derive the equation for the economic constraint in order to see how it changes with respect to e.g. consumption and the real interest rate, to see if it confirms our intuition that 1) the opportunity cost ought to rise (i.e., X_S as defined should fall) with consumption, as resources are committed at the expense of investment, and 2) the opportunity cost should rise with the real interest rate, as the return foregone increases.

To derive the economic constraint we utilize the following (Legendre) transformation: we define a function of state $A(p,c)$ such that

$$A(p,c) = \inf_X \left[p_y y(X,c) - \frac{1}{r} X \right]. \tag{6.3}$$

We know that the infimum is achieved when $X = X_S(p,c)$, i.e. at the economic constraint, because

$$\left(\frac{\partial y(X,c)}{\partial X} \right)_{X=X_S} = \frac{1}{p_y r} \tag{6.4}$$

when $X = X_S(p,c)$. Thus,

$$A(p,c) = p_y y(X_S(p,c)) - \frac{1}{r} X_S(p,c), \tag{6.5}$$

or

$$A(p,c) = p_y y(X_S(p,c)) - k_{c,p} \left(c^2 + p^2\right)^{1/2} X_S(p,c) \tag{6.6}$$

and

$$\begin{aligned} \left(\frac{\partial A(p,c)}{\partial p} \right)_c &= p_y \left(\frac{\partial y(X_S,c)}{\partial X_S} \right) \left(\frac{\partial X_S(p,c)}{\partial p} \right)_c - \frac{1}{2} k_{c,p} (c^2+p^2)^{-1/2} (-2p) X_S(p,c) \\ &\quad - k_{c,p} (c^2+p^2)^{1/2} \left(\frac{\partial X_S(p,c)}{\partial p} \right)_c, \end{aligned} \tag{6.7}$$

$$\begin{aligned} \left(\frac{\partial A(p,c)}{\partial c} \right)_p &= p_y \left(\frac{\partial y(X_S,c)}{\partial X_S} \right) \left(\frac{\partial X_S(p,c)}{\partial c} \right)_p + p_y \left(\frac{\partial y(X_S,c)}{\partial c} \right)_p - \frac{1}{2} k_{c,p} (c^2+p^2)^{-1/2} (-2c) X_S(p,c) \\ &\quad - k_{c,p} (c^2+p^2)^{1/2} \left(\frac{\partial X_S(p,c)}{\partial c} \right)_p. \end{aligned} \tag{6.8}$$

These relations can be considerably simplified by noting that

$$\left(\frac{\partial y(X_S,c)}{\partial X_S}\right)_c = \frac{1}{p_y r}, \tag{6.9}$$

$$\left(\frac{\partial y(X_S,c)}{\partial c}\right)_p = \frac{p}{p_y} \tag{6.10}$$

so that by substitution

$$\left(\frac{\partial A(p,c)}{\partial p}\right)_c = k_{c,p} p \left(c^2+p^2\right)^{-1/2} X_S(p,c), \tag{6.11}$$

$$\left(\frac{\partial A(p,c)}{\partial c}\right)_p = p + k_{c,p} c \left(c^2+p^2\right)^{-1/2} X_S(p,c). \tag{6.12}$$

Since $A(p,c)$ is a function of state we know that

$$\frac{\partial}{\partial c}\left(\frac{\partial A(p,c)}{\partial p}\right)_c = \frac{\partial}{\partial p}\left(\frac{\partial A(p,c)}{\partial c}\right)_p, \tag{6.13}$$

or (after simplifying)

$$p\left(c^2+p^2\right)^{-1/2}\frac{\partial X_S(p,c)}{\partial c} - c\left(c^2+p^2\right)^{-1/2}\frac{\partial X_S(p,c)}{\partial p} = \frac{1}{k_{c,p}}. \tag{6.14}$$

A closed-form family of solutions to this partial differential equation exists–it is

$$X_S(p,c) = \pm\left(c^2+p^2\right)^{1/2}\frac{1}{k_{c,p}}\arctan[\frac{p}{c}] + \Phi(c^2+p^2), \tag{6.15}$$

or

$$X_S(r,c) = \pm\frac{1}{(k_{cp})^2 r}\arctan\left[\frac{\left(\frac{1}{(k_{c,p}r)^2}-c^2\right)^{1/2}}{c}\right] + \Phi\left(\frac{1}{(k_{c,p}r)^2}\right) \tag{6.16}$$

Again for simplicity we shall consider the solutions for which $\Phi = 0$, and will select the positive root as it is this solution which agrees with our requirement that dy be an exact differential (this can be seen by finding the conditions under which mixed partial derivatives of y with respect to c and r are equal–see the income analysis below). We note immediately that with this solution

$$\left(\frac{\partial X_S(r,c)}{\partial c}\right)_r = -\frac{1}{p(k_{c,p})^2 r}, \tag{6.17}$$

$$\left(\frac{\partial X_S(r,c)}{\partial r}\right)_c = -\frac{1}{(k_{c,p} r)^2}\left(\frac{c}{p} + \text{arccot}[\frac{c}{p}]\right). \tag{6.18}$$

Since the state variables c, p, and r are all positive, both of these partial derivatives are negative–that is, the economic constraint (or opportunity cost) rises as either consumption rises (for a fixed interest rate) or the interest rate rises (for a fixed level of consumption). Our intuition is confirmed. In addition, it is to be noted that by far the greatest impact upon the economic constraint comes from the interest rate, due to the $\frac{1}{(k_{cp})^2 r}$ term in equation (6.16).

We next derive income as a function of state. We know that

$$\left(\frac{\partial y(r,c)}{\partial r}\right)_c = \left(\frac{\partial y(X_S,c)}{\partial X_S}\right)_c \left(\frac{\partial X_S(r,c)}{\partial r}\right)_c = \tag{6.19}$$

$$\frac{1}{p_y r}\left(-\frac{1}{(k_{c,p} r)^2}\left(\frac{c}{p} + \text{arccot}[\frac{c}{p}]\right)\right), \tag{6.20}$$

and

$$\left(\frac{\partial y(r,c)}{\partial c}\right)_r = \left(\frac{\partial y(X_S,c)}{\partial X_S}\right)_c \left(\frac{\partial X_S(r,c)}{\partial c}\right)_r + \left(\frac{\partial y(X_S,c)}{\partial c}\right)_{X_S} = \tag{6.21}$$

$$\frac{1}{p_y r}(-\frac{1}{p(k_{c,p})^2 r}) + \frac{p}{p_y}. \tag{6.22}$$

Note that (6.22) is equivalent to (5.14), since

$$\frac{1}{p_y r}(-\frac{1}{p(k_{c,p})^2 r}) + \frac{p}{p_y} = \frac{-(c^2+p^2)}{p_y p} + \frac{p}{p_y} = -\frac{c^2}{p_y p}. \tag{6.23}$$

This confirms that our selection of the positive root of (6.16) with $\Phi = 0$ is consistent with what we already know about income (further evidence is given in deriving the consistency conditions for the mixed partial derivatives in (6.26), which we omit here for brevity).

These partial derivatives for income again confirm our intuition that as consumption or the interest rate rises, income will fall. Although the exact mechanisms dictating these changes are left unspecified (for they belong to the realm of the underlying microeconomy), the effect upon income is nonetheless what we would expect.

We may now define the exact differential $dY = p_y dy$ with respect to c and r as

$$p_y dy = p_y\left(\frac{\partial y(r,c)}{\partial r}\right)_c dr + p_y\left(\frac{\partial y(r,c)}{\partial c}\right)_r dc = \tag{6.24}$$

$$\frac{1}{r}(-\frac{1}{(k_{c,p} r)^2}\left(\frac{c}{p} + \text{arccot}[\frac{c}{p}]\right) dr + \left(\frac{1}{r}(-\frac{1}{p(k_{c,p})^2 r}) + p\right) dc, \tag{6.25}$$

from which it may be verified that

$$\frac{\partial}{\partial c}\left(\frac{\partial y(r,c)}{\partial r}\right)_c = \frac{\partial}{\partial r}\left(\frac{\partial y(r,c)}{\partial c}\right)_r, \tag{6.26}$$

as expected.

Integrating (6.24) yields the solution (neglecting constants of integration):

$$Y(c,r) = p_y y(c,r) = \frac{1}{2}c\left(\frac{1}{(k_{c,p}r)^2} - c^2\right)^{1/2} - \frac{1}{2(k_{c,p}r)^2}\left(\arcsin(k_{c,p}rc) - \frac{\pi}{2}\right), \tag{6.27}$$

or

$$Y = \frac{1}{2}cp - \frac{1}{2(k_{c,p}r)^2}\left(\arcsin(k_{c,p}rc) - \frac{\pi}{2}\right). \tag{6.28}$$

Thus nominal income is half of the value of consumption adjusted for the interest rate. It is clear from this that a higher interest rate implies lower attainable values for p and c (from equation [4.30]) or equivalently a higher opportunity cost penalty. As with the economic constraint it is the interest rate which has the greatest effect, since changes in consumption or the price level only significantly change output as either of them approach zero. Lower interest rates imply a lower opportunity cost, higher possible levels of consumption, and consequently a higher level of income.

7 Concluding Remarks

The methodology introduced in this paper carries with it a host of implications which cannot be given in their entirety here–the outline sketched here is meant to simply 'whet the appetite'. Readers well-versed in the study of conservative systems, and primarily thermodynamics, will have no difficulty in following the arguments made, and are free to generate useful economic parallels to other quantities such as the thermodynamic potentials (one of which has already been used briefly here). But the point of the exercise has been to show that a few straightforward assumptions about the gross properties of any macroeconomy (namely, the conservation of income and the law of non-arbitrage) can generate a rich assortment of conclusions without having to place too much unnecessary or unrealistic structure upon its surface. The existence of the economic constraint and its deep relation to the concept of opportunity cost is one such benefit.

But the analysis, as such, is incomplete. We know that any macroeconomy is (at least) the sum of its microeconomic parts, and any treatment of the gross properties must at some point be reconciled with what is known about the microeconomy before stronger conclusions may be drawn. Such a reconciliation is a program for future research, in which it is to be shown that aggregate state variables are well-defined averages of known microeconomic activity. Such activity does not imply the existence of a representative agent with a well-defined utility function, although the underlying microeconomic agents may possess and use such functions. Rather, the macroeconomy is a summary of countless interactions which must be addressed at the statistical level so that the organizing properties of the macroeconomy are understood. In this light, the more complicated and assuredly more realistic non-conservative dynamical systems which appear to compose the majority of macroeconomic behavior may find a more fundamental and hence more palatable foundation.

8 References

Reiss, Howard. Methods of Thermodynamics. Dover Publications, 1965.

9 Figures

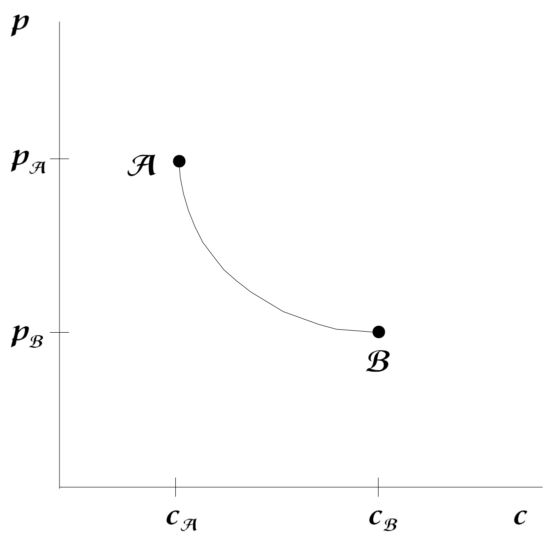

Figure 1: An Equilibrium Transition

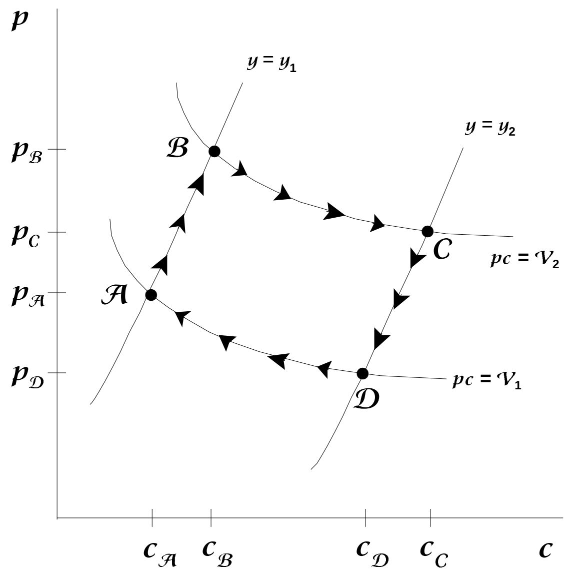

Figure 2: A Simple Business Cycle

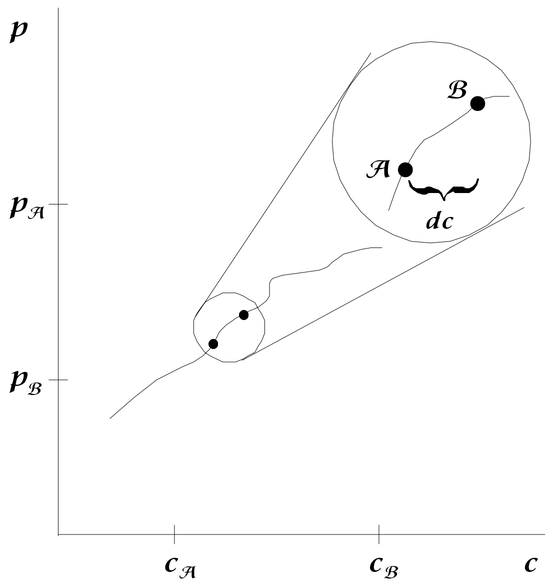

Figure 3: A Small Change in Consumption Over a Smooth Path

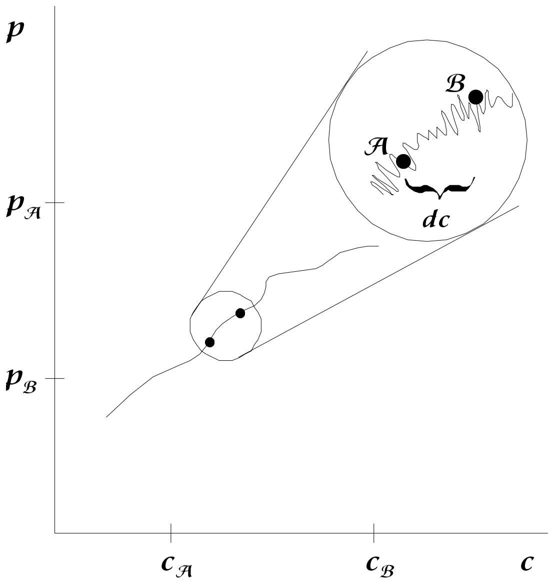

Figure 4: A Small Change in Consumption Over a Wiggly Path

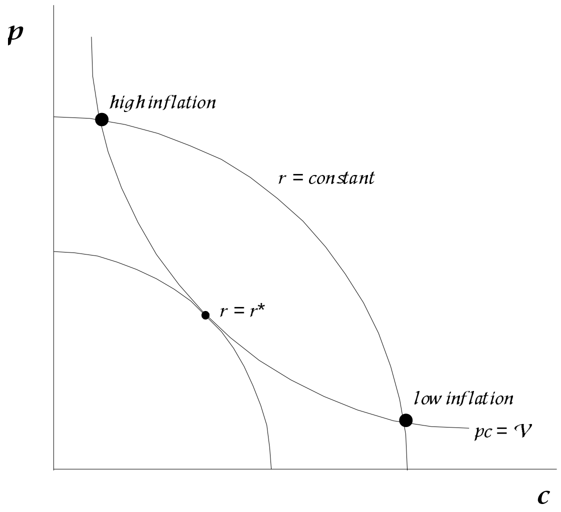

Figure 5: High and Low Inflationary Regimes

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