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The Bi-parameter Smooth Transition Autoregressive model*

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Abstract

The paper introduces a nonlinear model that belongs to the STAR family of models. The main feature of the suggested Bi-parameter Smooth Transition Autoregressive (BSTAR) model is that it allows for different speeds of transition between the middle regime and each of the identical outer regimes. Thus, the BSTAR model can be considered as a generalization of the LSTR2 model introduced in Teräsvirta (1998) which imposes symmetric speed of adjustment between the middle and each of the identical outer regimes.

Key words: STAR models; Nonlinear time series; Index of Industrial Production.

JEL Classification Code: C22, C50, E23

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1 Introduction.

While linear models have proved to be quite useful as a modelling device in both applied and theoretical sciences, such models have their limitations. Potentially, nonlinear models are naturally called for to overcome these inherent limitations. However, the price to be paid is increasing complexity of estimation and inference in nonlinear models and also such models can be too flexible to be powerfully rejected against alternative models. Nevertheless, the recent advances in computer technology makes the life of an applied scientist somewhat easier as regarding implementation and estimation of nonlinear models. This in turn gives rise to an increasing number of applications of nonlinear models in science more generally and in economics in particular, e.g. see Tong (1990) and Granger and Terasvirta (1993) for introduction to the fascinating world of non-linear time series models.

Several non-linear models have been suggested in the literature. The present paper confines itself to the class of non-linear models popularized by Timo Teräsvirta and co-authors in a series of articles over the recent decade. This research agenda is summarized in Teräsvirta (1994a), Teräsvirta (1998), and van Dijk, Teräsvirta, and Franses (2000).

In these articles the authors suggest the so-called Smooth Transition Threshold Autoregressive (STAR) models. The STAR type of models involve two, three or, possibly, more regimes such that the transition between the different regimes can occur rather smoothly. The STAR model family can be considered the generalization of the multiple regime Self-Exciting Threshold Autoregressive (SETAR) class of models where the transition between the different regimes takes place rather abruptly. The latter type of models approximate the underlying nonlinear process piecewise linearly such that the process is linear in each regime, see Tong (1990) for a presentation of the SETAR models.

The transition function in the STAR type of models plays the central role. As a rule the transition function has two types of parameters, namely, the slope and the threshold parameters. The former parameter denotes the speed of the transition between the different regimes whereas the latter denotes the location of the transition from one regime to another.

Teräsvirta (1994b) proposed two types of STAR models depending on whether a logistic or an exponential transition function is used. Note that both these transition functions posses only one slope and threshold parameter. The former corresponds to the so-called LSTR1 model with two regimes and the latter forms the ESTAR model with three regimes in such a way that the two outer regimes becomes identical. The drawback of the latter transition function is that both for small and large slope parameter values the nonlinearity practically disappears. To correct for this unfortunate fact Teräsvirta (1998) suggests to replace the transition function of the ESTAR model with another transition function whereby for large values of the slope parameter the nonlinear model is no longer indistinguishable from the linear model. This is achieved by introducing the second threshold parameter. The resulting model with a new transition function has been denoted the LSTR2 model.

The ESTAR model of Teräsvirta (1994b) involves a transition function that is symmetric around the threshold value. Anderson (1997) extends the exponential transition function to be asymmetric around the threshold value, henceforth referred as the AESTAR model. However, this asymmetric transition function suffers from the same drawback as the symmetric one in the ESTAR model. So far no corresponding counterpart of the LSTR2 model has been suggested which allows for different speeds of transition between the outer and middle regimes.

The contribution of the present paper is to fill this gap in the literature, and in so doing it proposes a STAR family model with a transition function that allows for different transition speeds between the middle and outer regimes. This is achieved by incorporating an additional slope parameter in the transition function. Thus, the suggested transition function has two threshold parameters and two slope parameters. Henceforth, the model in question is referred to as the Bi-parameter Smooth Transition Autoregressive model (BSTAR in short).

The paper proceeds as follows. Section 2 provides an overview of the STAR family of models and Section 3 introduces the BSTAR model. The estimation of the BSTAR model by means of the maximum likelihood method is discussed in Section 4. Section 5 derives the Lagrange Multiplier test for linearity against the BSTAR model. Application of the misspecification tests suggested in Eirtheim and Teräsvirta (1996) to the BSTAR model is discussed in Section 6. Section 7 contains an empirical example. The final section concludes.

All computations were performed using the object-oriented programming language Ox 2.20 Professional, see Doornik (1999), and the empirical modelling program package PcGive 9.1, see Hendry and Doornik (1999).

2 Literature review.

In general, the scalar univariate STAR(p) family of models looks like

$$y_t = \boldsymbol{\phi}'\mathbf{x}_t + \boldsymbol{\theta}'\mathbf{x}_t F_t\left(y_{t-d}\right) + u_t \quad , \tag{1}$$

where $\mathbf{x}_t = \{1, y_{t-1}, ..., y_{t-p}\}'$ is a vector of the lags of the dependent variable including the constant term. The vectors of the autoregressive parameters are $\boldsymbol{\phi} = \left(\phi_0, \phi_1, ..., \phi_p\right)'$ and $\boldsymbol{\theta} = \left(\theta_0, \theta_1, ..., \theta_p\right)'$. The error term u_t is usually assumed to be the $\mathcal{NID}$ random variable with mean zero and variance σ^2. The transition function $F_t\left(y_{t-d}\right)$ can take different forms discussed below in more detail.

The natural starting point in describing the STAR model family is the two-regime LSTR1 model with the following logistic transition function, which takes values in the interval between zero and one:

$$F_t\left(\gamma, c; y_{t-d}\right) = \frac{1}{1 + \exp\left(-\gamma\left(y_{t-d} - c\right)\right)} \quad , \ \gamma > 0. \tag{2}$$

At this stage, it is convenient to introduce some terminology for further reference. The transition function (2) has several parameters such as the slope

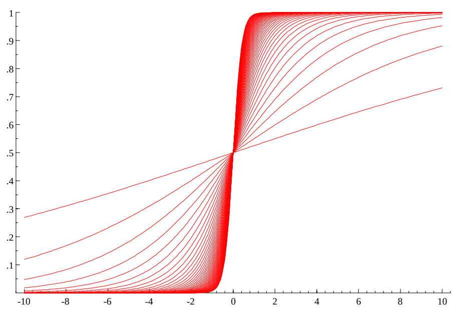

Figure 1: LSTR1 model; transition function (2) with slope parameter $\gamma = [0.1, 5]$ and step 0.1, and with location parameter $c = 0$.

parameter γ, the threshold parameter c, and the delay parameter d. The magnitude of the slope parameter measures the speed of transition between the two regimes while the value of the threshold parameter indicates the location of the transition. The delay parameter, which appears in connection with the transition variable y_{t-d}, can take values that are either less or equal, $0 < d \leq p$, or greater, $p < d$, than the order of the STAR model.[1]

Figure 1 presents an example of the LSTR1 transition function for different slope parameter values, $\gamma = [0.1, 5]$. Note that as $\gamma \to 0$, the transition function tends to a constant value of 0.5 such that the LSTR1 model becomes linear. On the other hand, the larger the value of γ, the steeper is the slope of the transition function (2) at the point $y_{t-d} = c$. As $\gamma \to \infty$ the LSTR1 transition function approaches the step function. Therefore, in the limit LSTR1 turns into a two-regime Self-Exciting Threshold Autoregressive (SETAR) model, (see Tong (1990) p.101).

The other possible choice of the transition function is given by the exponential transition function:

$$F_t(\gamma, c; y_{t-d}) = 1 - \exp\left(-\gamma (y_{t-d} - c)^2\right) \quad , \quad \gamma > 0. \tag{3}$$

The ESTAR model is a three-regime model that comprises one middle regime and two equivalent outer regimes. The transition function of the ESTAR model is depicted in Figure 2 with the slope parameter γ taking values in the interval $[0.1, 5]$. Notice that on the one hand the ESTAR model is practically linear when $\gamma \to 0$. On the other hand, as $\gamma \to \infty$, the transition function is equal to unity elsewhere except at the threshold point c, where it equals zero. Thus for large values of γ the ESTAR model becomes practically indistinguishable from the linear model as well. This also implies that the three-regime SETAR model does not constitute a special case of the ESTAR model.

[1]Moreover, the transition variable can be any other variable or even a linear combination of several variables rather than own lags of the endogenous variable, y_{t-d}. Similarly, other exogenous variables can be used in (1) as the explanatory variables.

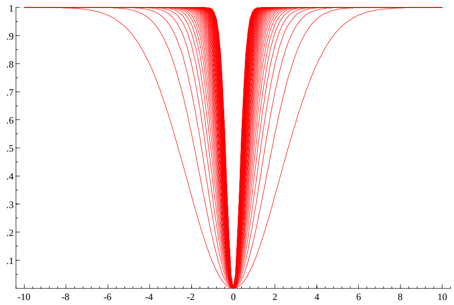

Figure 2: ESTAR model; transition function (3) with slope parameter $\gamma = [0.1, 5]$ and step 0.1, and with location parameter $c = 0$.

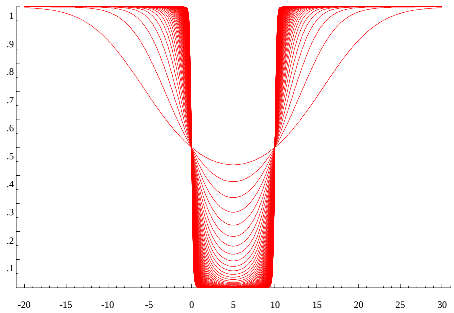

Figure 3: LSTR2 model; transition function (4) with slope parameter $\gamma = [0.01, 1]$ and step 0.01, and with location parameters $c_1 = 0$ and $c_2 = 10$.

If such a limiting behavior of the ESTAR model is undesirable, i.e. for $\gamma \to \infty$, then one can resort to the LSTR2 model suggested in Teräsvirta (1998). The transition function of the LSTR2 model is the second-order logistic function:

$$F_t(\gamma, c_1, c_2; y_{t-d}) = \frac{1}{1 + \exp\left(-\gamma\left(y_{t-d} - c_1\right)\left(y_{t-d} - c_2\right)\right)}\ , \quad \gamma > 0. \tag{4}$$

Figure 3 depicts this transition function for different values of the slope parameter $\gamma = [0.01, 1]$. The LSTR2 transition function resembles the transition function of the LSTR1 model but it involves two threshold parameters c_1 and c_2. From Figure 3 it is seen that as $\gamma \to \infty$ the transition function no longer collapses to a point but becomes a three-regime SETAR model. Therefore, the LSTR2 model is free from the drawback of the ESTAR model as $\gamma \to \infty$. At the same time, however, the range of the transition function depends on the value of the slope parameter.

The three-regime ESTAR and LSTAR2 models considered above are char-

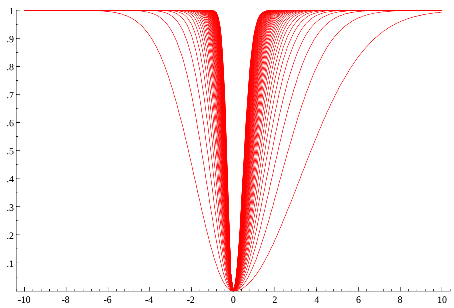

Figure 4: AESTAR model; transition function (5) with slope parameter $\gamma = [0.1, 5]$ and 0.1 step, and with asymmetry parameter $\delta = -17$, and with location parameter $c = 0$.

acterized by their respective symmetric transition functions which allow for the same transition speed between the outer-lower and middle regime on the one hand, and on the other hand between the middle and outer-higher regime, as seen in Figures 2 and 3. In some cases, this might be too restrictive an assumption.

To circumvent this property Anderson (1997) modifies the ESTAR transition function (3) in such a way that it becomes asymmetric around the location parameter value c. Henceforth the model with the asymmetric transition function given below is correspondingly referred as to the Asymmetric ESTAR model (AESTAR in short):

$$\begin{aligned} F_t(\gamma, c, \delta; y_{t-d}) &= 1 - \exp\{-\gamma [y_{t-d} - c]^2 \times h(y_{t-d})\}, \ \gamma > 0, \\ h_t(c, \delta; y_{t-d}) &= \{0.5 + (1 + \exp\{-\delta [y_{t-d} - c])^{-1}\}, \ \delta \neq 0. \end{aligned} \tag{5}$$

Notice that positive or negative values of the parameter δ introduce asymmetry in the transition function given in (5), see Figures 4 and 5. Hence the AESTAR model allows for different speeds of transition between the middle regime and each of its outer regimes.

Figures 4 and 5 reveal the fact that the AESTAR model entails the same problem as the ESTAR model, i.e. for large values of γ it becomes difficult to distinguish it from a linear model. Hence the three-regime SETAR model is not nested within the AESTAR model either.

The next section introduces the BSTAR model which extends the LSTR2 model to allow for asymmetric speed of transition between the outer and middle regimes, similarly as the model of Anderson (1997) extends the ESTAR model.

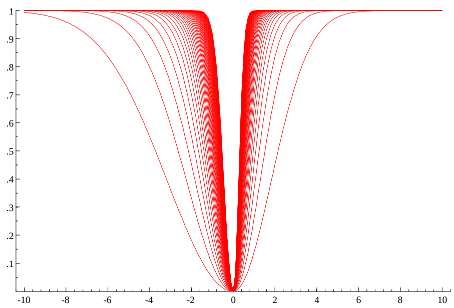

Figure 5: AESTAR model; transition function (5) with slope parameter $\gamma = [0.1, 5]$ and 0.1 step, and with the asymmetry parameter $\delta = 17$, and with location parameter $c = 0$.

3 BSTAR model.

It was stated earlier that the asymmetric transition function of Anderson (1997) provides an extension of the transition function of the ESTAR model. Therefore it has a similar inherent problem when the slope parameter tends to infinity: it is problematic to distinguish this non-linear model from a linear one.

In this section, the following transition function is introduced:

$$F_t(\gamma_1, c_1, \gamma_2, c_2; y_{t-d}) = \frac{\exp[-\gamma_1(y_{t-d} - c_1)] + \exp[\gamma_2(y_{t-d} - c_2)]}{1 + \exp[-\gamma_1(y_{t-d} - c_1)] + \exp[\gamma_2(y_{t-d} - c_2)]} \quad (6)$$

$$\gamma_1 > 0, \; \gamma_2 > 0, \; c_1 < c_2.$$

This transition function has two threshold parameters c_1 and c_2, and two slope parameters γ_1 and γ_2. The latter parameters are, in general, different and therefore the slopes of the transition function at the two threshold parameters c_1 and c_2 are, in general, different in the BSTAR model. This allows for asymmetric speed of transition from the outer-lower regime to the middle one and from the middle one to the outer-higher one. Thus it serves as a natural extension of the LSTR2 model to allow for asymmetric transition similar to the AESTAR model of Anderson (1997), which itself is an extension of the ESTAR model. In this way the problems associated with very large slope parameter values in the (A)ESTAR models are avoided.

Notice that the transition function (6) offers a more expository way to capture the asymmetry in the speed of transition when it is compared to the approach of Anderson (1997) and embodied in the transition function of the AESTAR model. In the transition function of the BSTAR model each slope parameter determines the steepness of the transition function at the corresponding transition location. Whereas in the AESTAR transition function (5) the only slope parameter determines the overall steepness of the transition function such

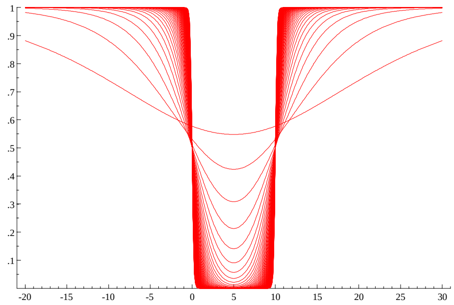

Figure 6: BSTAR model; transition function (6) with slope parameters $\gamma_1 = [0.1, 10]$, $\gamma_2 = [0.1, 10]$ with step 0.1, restriction $\gamma_1 = \gamma_2$, and with location parameters $c_1 = 0, c_2 = 10$.

that the asymmetry of the AESTAR transition function is the result of the interplay between the slope parameter γ and the asymmetry parameter δ.

Figures 6, 7, and 8 illustrate the possible shapes of the transition function. It is seen that as long as the two threshold parameters c_1 and c_2 are different, the BSTAR transition function no longer collapses to a point. On the contrary, the magnitude of each slope parameter determines the steepness of the corresponding slope of the transition function and thus the corresponding speed of transition between the outer and middle regimes. Also notice that the BSTAR transition function offers greater shape variety than the AESTAR transition function. To see this, compare Figures 7, 8 and 4, 5.

Note that when the restriction of equality of the slope parameters is imposed, i.e. $\gamma_1 = \gamma_2 = \gamma$, the BSTAR transition function closely approximates the LSTR2 transition function, especially for large values of the slope parameter as seen at Figure 6.

Given such a large variety of shapes of the transition function of the BSTAR model and its close resemblance to the transition functions of the other models suggested in the literature, it seems that the BSTAR model could serve as an interesting alternative to these models.

One remark deserving attention regards the relationship between the BSTAR model and the Multiple Regime STAR (MRSTAR) model suggested in van Dijk (1999). The MRSTAR model can potentially comprise more than two regimes. This is achieved by adding the logistic transition functions, each of which has its own location and slope parameters, such that, for example, the MRSTAR model can be written as

$$y_t = \phi' x_t + \sum_{i=1}^{m} \theta_i' x_t G_i \left(\gamma_i, c_i; y_{t-d}\right) + \varepsilon_t,$$

where it is assumed that all the m transition functions depend on the same transition variable y_{t-d}.

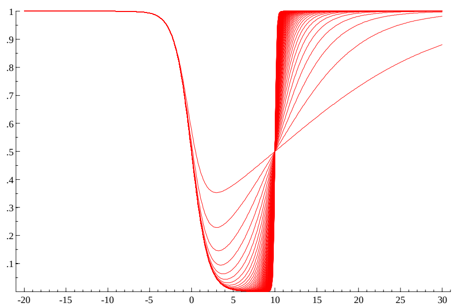

Figure 7: BSTAR model; transition function (6) with slope parameters $\gamma_1 = 1$, $\gamma_2 = [0.1, 10]$ and step 0.1, and with location parameters $c_1 = 0, c_2 = 10$.

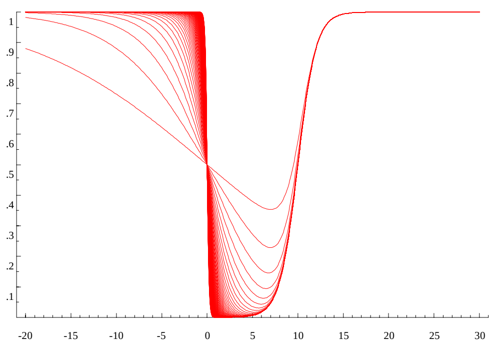

Figure 8: BSTAR model; transition function (6) with slope parameters $\gamma_1 = [0.1, 10]$ and step 0.1, $\gamma_2 = 1$, and with location parameters $c_1 = 0, c_2 = 10$.

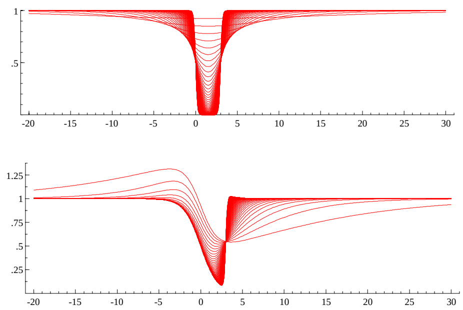

Figure 9: MRSTAR model (8). Transition function with location parameters $c_1 = 0$ and $c_2 = 2.5$. Upper panel: slope parameters $\gamma_1 = \gamma_2 = [0.1, 10]$ and with step 0.1. Lower panel: slope parameters $\gamma_1 = 1$, $\gamma_2 = [0.1, 10]$ and with step 0.1.

This approach offers great flexibility of the resulting transition function. In fact, I would argue, too much flexibility for the specific case of the three-regime LSTR2, ESTAR, and BSTAR models with one middle and two equivalent outer regimes as considered here. To illustrate this point, consider the case when $m = 2$, such that the MRSTAR model involves three regimes:

$$y_t = \phi' x_t + \theta_1' x_t G_1 (\gamma_1, c_1; y_{t-d}) + \theta_2' x_t G_2 (\gamma_2, c_2; y_{t-d}) + \varepsilon_t. \tag{7}$$

Furthermore, assume that the parameter vector is such that $\theta_1 = \theta_2 = \theta^*$. The transition functions $G_1 (\gamma_1, c_1; y_{t-d})$ and $G_2 (\gamma_2, c_2; y_{t-d})$ are as follows

$$\begin{aligned} G_1 (\gamma_1, c_1; y_{t-d}) &= \frac{1}{1 + \exp(\ \gamma_1 (y_{t-d} - c_1))} , \\ G_2 (\gamma_2, c_2; y_{t-d}) &= \frac{1}{1 + \exp(-\gamma_2 (y_{t-d} - c_2))} , \end{aligned}$$

where $\gamma_1 > 0$, $\gamma_2 > 0$, and $c_1 < c_2$. Then model (7) simplifies to

$$y_t = \phi' x_t + \theta^{*\prime} x_t [G_1 (\gamma_1, c_1; y_{t-d}) + G_2 (\gamma_2, c_2; y_{t-d})] + \varepsilon_t. \tag{8}$$

The upper panel of Figure 9 displays possible shapes of the transition function under the restriction of equal slope coefficients. Note that when the slopes parameters γ_1 and γ_2 are equal, the resulting transition function resembles the transition function of either the BSTAR with equal slope coefficients or the LSTR2 model.

The lower panel of Figure 9 displays the resulting transition function when the slope coefficients are allowed to differ. Potentially, this transition function is expected to reflect a three-regime model with different transition speeds between the middle and each of the equivalent outer regimes. As is seen, this is not quite the case, as the range of the transition function is no longer bounded to be in the interval between zero and one as a result of humps that appear around the location parameters c_1 and c_2. Thus, in this specific case of a three-regime model with two identical outer regimes and with asymmetric speed of transition, the greater flexibility of the MRSTAR transition function constitutes its slight drawback. This implies that in this situation the BSTAR models should be preferred to the MRSTAR model.

4 ML estimation.

In this section the question of estimation of the BSTAR model using the maximum likelihood method is addressed. Below the gradient vector under the assumption of a normally distributed error term is derived. This is used to calculate the BHHH approximation to the information matrix. The variance-covariance matrix of the parameter estimators is calculated in the usual way using the inverted information matrix.

To recall, the BSTAR(p) model is defined as

$$\begin{aligned} y_t &= \boldsymbol{\phi}'\mathbf{x}_t + \left(\boldsymbol{\theta}'\mathbf{x}_t\right) F_t\left(y_{t-d}\right) + u_t \quad u_t \sim \mathcal{NID}\left(0, \sigma^2\right) \quad , \\ F_t\left(y_{t-d}\right) &= \frac{\exp\left(-\frac{\gamma_1[y_{t-d}-c_1]}{sc}\right) + \exp\left(\frac{\gamma_2[y_{t-d}-c_2]}{sc}\right)}{1+\exp\left(-\frac{\gamma_1[y_{t-d}-c_1]}{sc}\right) + \exp\left(\frac{\gamma_2[y_{t-d}-c_2]}{sc}\right)} \quad , \\ \gamma_1 &> 0,\ \gamma_2 > 0,\ c_1 < c_2, \end{aligned}$$

where $\mathbf{x}_t = \{1, y_{t-1}, ..., y_{t-p}\}'$ is a vector of the lags of the dependent variable including the constant term. The vectors of the autoregressive parameters are $\boldsymbol{\phi} = \left(\phi_0, \phi_1, ..., \phi_p\right)'$ and $\boldsymbol{\theta} = \left(\theta_0, \theta_1, ..., \theta_p\right)'$.

Notice that the values of the slope parameters γ_1 and γ_2 are not independent of the values that are taken by the transition variable y_{t-d}. Teräsvirta (1994a) suggests to standardize the transition variable in the usual way by dividing it by scaling factor sc, which is equal to the sample standard deviation of the transition variable. Such scaling is necessary in order to make the slope parameter scale free from the variance of the transition variable. Teräsvirta (1998) mentions two advantages of such scaling. First, there is an advantage in numerical estimation, as scaling may bring the slope parameters to the similar magnitude of the other parameters in the model - unless the slope parameters are very large. Second, such scaling may facilitate the search for initial values of the slope parameters.

The likelihood function of the model, assuming the $\mathcal{NID}$ nature of the error term, is

$$L = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{e_t^2}{2\sigma^2}\right).$$

The log-likelihood of one observation, $\ln l_t$, for $t = 1, ...T$ is

$$\ln l_t = -\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\left[y_t - \boldsymbol{\phi}'\mathbf{x}_t - \left(\boldsymbol{\theta}'\mathbf{x}_t\right)F_t\left(y_{t-d}\right)\right]^2.$$

The partial derivatives of $\ln l_t$ with respect to the model parameters are

$$\begin{aligned}\frac{\partial \ln l_t}{\partial \boldsymbol{\phi}} &= \frac{1}{\sigma^2}e_t\mathbf{x}_t \\ \frac{\partial \ln l_t}{\partial \boldsymbol{\theta}} &= \frac{1}{\sigma^2}e_t\mathbf{x}_t F_t\left(y_{t-d}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial \ln l_t}{\partial \gamma_1} &= -\frac{1}{\sigma^2}e_t\left(\boldsymbol{\theta}'\mathbf{x}_t\right)\frac{\left(y_{t-d}-c_1\right)}{sc}\exp\left(-\frac{\gamma_1\left[y_{t-d}-c_1\right]}{sc}\right)\left[1-F_t\left(y_{t-d}\right)\right]^2 \\ \frac{\partial \ln l_t}{\partial c_1} &= \frac{1}{\sigma^2}e_t\left(\boldsymbol{\theta}'\mathbf{x}_t\right)\frac{\gamma_1}{sc}\exp\left(-\frac{\gamma_1\left[y_{t-d}-c_1\right]}{sc}\right)\left[1-F_t\left(y_{t-d}\right)\right]^2 \\ \frac{\partial \ln l_t}{\partial \gamma_2} &= \frac{1}{\sigma^2}e_t\left(\boldsymbol{\theta}'\mathbf{x}_t\right)\frac{\left(y_{t-d}-c_2\right)}{sc}\exp\left(\frac{\gamma_2\left[y_{t-d}-c_2\right]}{sc}\right)\left[1-F_t\left(y_{t-d}\right)\right]^2 \\ \frac{\partial \ln l_t}{\partial c_2} &= -\frac{1}{\sigma^2}e_t\left(\boldsymbol{\theta}'\mathbf{x}_t\right)\frac{\gamma_2}{sc}\exp\left(\frac{\gamma_2\left[y_{t-d}-c_2\right]}{sc}\right)\left[1-F_t\left(y_{t-d}\right)\right]^2.\end{aligned}$$

Notice that in deriving the score vector the variance of the error term σ^2 is assumed fixed. This is because the second partial derivative with respect to the parameter σ^2 forms its own block such that the resulting information matrix is block-diagonal. The assumption of treating the parameter σ^2 fixed in deriving the score vector is similarly made, for example, in Eirtheim and Teräsvirta (1996).

The maximum likelihood parameter values are obtained in the usual way as a solution to the likelihood equations using numerical optimization methods. The standard errors are computed by inverting the BHHH approximation[2] to the information matrix evaluated at the maximum likelihood parameter estimates

$$I^{-1}\left(\widehat{\phi},\widehat{\theta},\widehat{\gamma}_1,\widehat{c}_1,\widehat{\gamma}_2,\widehat{c}_2\right) = \left[\sum_{t=1}^{T}\widehat{g}_t\widehat{g}_t'\right]^{-1},$$

where $\widehat{g}_t = \{\frac{\partial \ln \widehat{l}_t}{\partial \phi'}, \frac{\partial \ln \widehat{l}_t}{\partial \theta'}, \frac{\partial \ln \widehat{l}_t}{\partial \gamma_1}, \frac{\partial \ln \widehat{l}_t}{\partial c_1}, \frac{\partial \ln \widehat{l}_t}{\partial \gamma_2}, \frac{\partial \ln \widehat{l}_t}{\partial c_2}\}'$ is a vector of dimension conformable with the number of the estimated parameters in the model.

In small samples, when the estimated transition function is close to being a step function, the estimated information matrix may appear singular and hence noninvertible. To understand why this might happen consider the partial

[2] Alternatively, one can approximate the information matrix with either analytical or numerical Hessian evaluated at the ML estimates. However, as presented below the BHHH approximation nicely illustrates what causes the calculated information matrix to be reported as being singular.

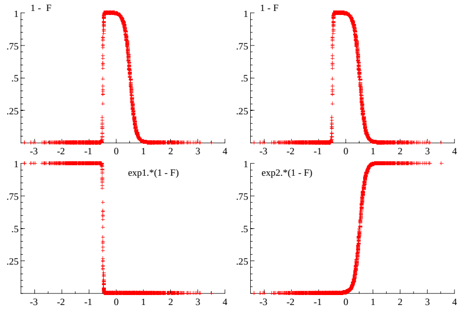

Figure 10: BSTAR model. Upper panels: terms $[1 - F_t(y_{t-d})]$, lower panels: terms $\left(\exp\left(-\frac{\gamma_1[y_{t-d}-c_1]}{sc}\right)[1 - F_t(y_{t-d})]\right)$, and $\left(\exp\left(\frac{\gamma_2[y_{t-d}-c_2]}{sc}\right)[1 - F_t(y_{t-d})]\right)$, $d = 1$, $\gamma_1 = 100$, $c_1 = -0.5$, $\gamma_2 = 10$, $c_2 = 0.5$, $T = 2000$.

derivatives of the log-likelihood of one observation with respect to γ_i and c_i. In these expressions, the following terms are of interest:

$$\exp\left(-\frac{\gamma_1[y_{t-d}-c_1]}{sc}\right)[1 - F_t(y_{t-d})]^2 \tag{9}$$

for γ_1 and c_1, and

$$\exp\left(\frac{\gamma_2[y_{t-d}-c_2]}{sc}\right)[1 - F_t(y_{t-d})]^2 \tag{10}$$

for γ_2 and c_2.

Figures 10 and 11 display these terms. Figures 10 and 11 were generated using artificial data with the following parameters of the transition function: $\gamma_1 = 100$, $c_1 = -0.5$, $\gamma_2 = 10$, $c_2 = 0.5$ for the sample size of 2000 observations. Notice that the slope parameters are allowed to differ such that the transition function has rather steep slope at the first location parameter value $c_1 = -0.5$ and somewhat lesser slope at the second location parameter value $c_2 = 0.5$.

The terms given in (9) and (10) are the multiplication product of the terms depicted at the corresponding upper and lower panels in Figure 10. The lower panels of Figure 11 depict the results of such multiplication. Observe that the only nonzero values of expression (9) are those that lie in the ultimate neighborhood of the first location parameter value. By contrast, the nonzero values of expression (10) are more spread out in the neighborhood of the second location parameter.

Thus, for a small sample size and large values of the slope parameter γ_i $i = 1, 2$, there might only be a few observations that lie in the ultimate neighborhood of the threshold. Consequently, in such a situation there might be

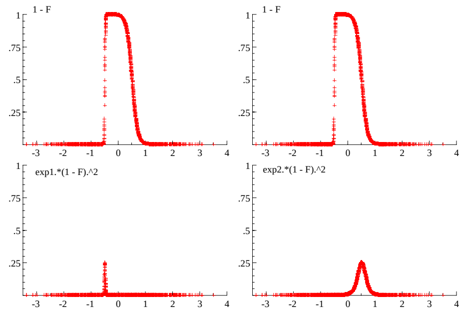

Figure 11: BSTAR model. Upper panels: terms $[1 - F_t(y_{t-d})]$, Lower panels: terms (9) and (10) . $d = 1$, $\gamma_1 = 100$, $c_1 = -0.5$, $\gamma_2 = 10$, $c_2 = 0.5$, $T = 2000$.

rather few observations that have a corresponding nonzero value for the respective partial derivative expression. This, in turn, implies that the calculated information matrix might be reported as being singular and therefore impossible to invert due to an insufficient number of observations in the neighborhood of the threshold parameters. To solve the problem of a singular information matrix Teräsvirta (1998) suggests that one omits the partial derivatives of the individual log-likelihoods with respect to those slope and location parameters that cause the problem.

5 LM-type test against BSTAR model.

Estimation of nonlinear models is more complex when compared to estimation of linear models. As a result one needs to assess whether it is worth spending extra time and efforts to fit a nonlinear model when much simpler linear models are available. This section derives the LM-type linearity test against the BSTAR model, both when the slope parameters are different and when they are equivalent. The approach undertaken here is similar to the derivation of linearity tests against the other STAR family models suggested in Teräsvirta (1994a), for example.

The BSTAR(p) model is

$$y_t = \boldsymbol{\phi}'\mathbf{x}_t + \boldsymbol{\theta}'\mathbf{x}_t F_t(y_{t-d}) + u_t \quad , \tag{11}$$

where $\mathbf{x}_t = \{1, y_{t-1}, ..., y_{t-p}\}'$ is a vector of the lags of the dependent variable including the constant term. For later use, define $\widetilde{\mathbf{x}}_t = \{y_{t-1}, ..., y_{t-p}\}'$ as a vector of the lags of the dependent variable. The vectors of the autoregressive parameters are $\boldsymbol{\phi} = (\phi_0, \phi_1, ..., \phi_p)'$ and $\boldsymbol{\theta} = (\theta_0, \theta_1, ..., \theta_p)'$. The error term u_t is assumed to be an $\mathcal{NID}$ random variable with variance σ^2. The corresponding transition function is

$$\begin{aligned} F_t(\gamma_1, c_1, \gamma_2, c_2; y_{t-d}) &= \frac{\exp[-\gamma_1(y_{t-d} - c_1)] + \exp[\gamma_2(y_{t-d} - c_2)]}{1 + \exp[-\gamma_1(y_{t-d} - c_1)] + \exp[\gamma_2(y_{t-d} - c_2)]} \\ \gamma_1 &> 0, \gamma_2 > 0, c_1 < c_2. \end{aligned}$$

First, consider the BSTAR model when the slope parameters are allowed to differ. The model given in (11) is linear when both the slope parameters are equal to zero $\gamma_1 = \gamma_2 = 0$ and/or when the parameter vector $\boldsymbol{\theta}$ is equal to zero $\boldsymbol{\theta} = 0$. This means that under the null hypothesis of linearity the model is not identified. This identification problem is similarly relevant to the other STAR family models. Therefore, the method for circumventing this problem is the same as when testing for linearity against the other STAR family models - see, for example, Saikkonen and Luukkonen (1988) and Luukkonen, Saikkonen, and Terasvirta (1988). That is, one should consider a Taylor series expansion of the transition function around the point $\gamma_1 = \gamma_2 = 0$.

In the following I will work with the redefined transition function $F_t^*(\cdot) = F_t(\cdot) - 2/3$, such that it takes a value of zero under the null hypothesis of linearity, such that the resulting linear model reads

$$y_t = \boldsymbol{\phi}'\mathbf{x}_t + u_t.$$

The first-order Taylor series expansion of the transition function around the point $\gamma_1 = \gamma_2 = 0$ yields

$$\begin{aligned} T_1 &= F_t^*(\cdot)|_{\gamma_1=\gamma_2=0} + \frac{\partial F_t^*(\cdot)}{\partial \gamma_1}|_{\gamma_1=\gamma_2=0} * \gamma_1 + \\ &\quad + \frac{\partial F_t^*(\cdot)}{\partial \gamma_2}|_{\gamma_1=\gamma_2=0} * \gamma_2 + R_1(\gamma_1, \gamma_2, c_1, c_2; y_{t-d}) \\ &= (-\frac{1}{9} * y_{t-d} + \frac{1}{9} * c_1) * \gamma_1 + (\frac{1}{9} * y_{t-d} - \frac{1}{9} * c_2) * \gamma_2 \\ &\quad + R_1(\gamma_1, \gamma_2, c_1, c_2; y_{t-d}). \end{aligned}$$

After substituting the Taylor series approximation into the regression equation (11) and rearranging the terms we get

$$y_t = \boldsymbol{\beta}_1'\mathbf{x}_t + \boldsymbol{\beta}_2'\mathbf{x}_t \cdot y_{t-d} + e_t, \tag{12}$$

where $e_t = \boldsymbol{\theta}'\mathbf{x}_t R_1 + u_t$ and

$$\begin{aligned} \boldsymbol{\beta}_1 &= \boldsymbol{\phi} + \frac{1}{9}\gamma_1 c_1 \boldsymbol{\theta} - \frac{1}{9}\gamma_2 c_2 \boldsymbol{\theta} \\ \boldsymbol{\beta}_2 &= \frac{1}{9}\boldsymbol{\theta}(\gamma_2 - \gamma_1). \end{aligned}$$

Equation (12) represents the auxiliary test regression for the case when the transition variable y_{t-d} is not an element of the vector $\widetilde{\mathbf{x}}_t = \{y_{t-1}, ..., y_{t-p}\}'$. This happens when the order of the BSTAR model is less than the delay parameter value, i.e. $p < d$.

From the coefficient expressions it is seen that the null hypothesis of linearity $\gamma_1 = \gamma_2 = 0$ corresponds to the condition that the parameter vector $\boldsymbol{\beta}_2$ equals zero, i.e. $\boldsymbol{\beta}_2 = 0$. Also notice that under the null hypothesis we have $e_t = u_t$, as the remaining term in the Taylor expansion $R_1(\gamma_1, \gamma_2, c_1, c_2; y_{t-d})$ equals zero. Given this then, linearity testing amounts to testing for insignificance of the parameter vector $\boldsymbol{\beta}_2$, a procedure which can be carried out in a straightforward manner by means of a LM-type test under the assumption of normality of the error term. Under the regularity conditions discussed in Luukkonen, Saikkonen, and Terasvirta (1988), the test statistic is $\chi^2(p+1)$ - distributed under the null hypothesis.

In the case the transition variable y_{t-d} corresponds to an element of the vector $\widetilde{\mathbf{x}}_t$ the auxiliary regression (12) needs be rearranged as

$$y_t = \beta_{1,0} + \widetilde{\boldsymbol{\beta}}_1' \widetilde{\mathbf{x}}_t + \beta_{2,0} y_{t-d} + \widetilde{\boldsymbol{\beta}}_2' \widetilde{\mathbf{x}}_t \cdot y_{t-d} + e_t \quad ,$$

such that the auxiliary regression takes the form

$$y_t = \beta_{1,0} + \widetilde{\widetilde{\boldsymbol{\beta}}}_1' \widetilde{\mathbf{x}}_t + \widetilde{\boldsymbol{\beta}}_2' \widetilde{\mathbf{x}}_t \cdot y_{t-d} + e_t, \tag{13}$$

with $\widetilde{\boldsymbol{\beta}}_j = \{\beta_{j,1}, ..., \beta_{j,p}\}'$ for $j = 1, 2$, and

$$\widetilde{\widetilde{\beta}}_{1,i} = \left\{ \begin{array}{cc} \widetilde{\beta}_{1,i} & if \;\; i \neq d \\ \widetilde{\beta}_{1,i} + \beta_{2,0} & if \;\; i = d \end{array} \right\} \; i = 1, ..., p.$$

There is one more remark regarding the use of regression equation (13) for linearity testing. The parameter θ_0 does not enter into the expressions for the parameter vector of interest $\widetilde{\boldsymbol{\beta}}_2$. This implies that the test based on the regression equation is likely to have rather low power when the intercept is the only parameter that is changing across the different regimes. See the discussion in Teräsvirta (1994a) of the similar problem that arises in the LSTR1 model when the auxiliary test equation is based on the first-order Taylor series expansion of the logistic transition function.

It is interesting to note that the condition of the slope coefficient equality, $\gamma_1 = \gamma_2$, also implies $\boldsymbol{\beta}_2 = 0$ in equation (12) and $\widetilde{\boldsymbol{\beta}}_2 = 0$ in equation (13). This in turn implies that the linearity test based on regressions (12) and (13) fails to distinguish between a linear underlying model and a nonlinear underlying model which is characterized by a BSTAR transition function with equal slope parameters. Hence, it would be not advisable to use regression equations (12) and (13) for linearity testing when one believes that the underlying model is the BSTAR model with equal slope coefficients.

This result is interesting to compare with the auxiliary regression that is used to test for linearity against the ESTAR and LSTR2 models because these models closely resemble the BSTAR model with equal slope coefficients. Indeed, when testing linearity against the ESTAR or LSTR2 models, the corresponding auxiliary test regressions have additional regressors such as $\mathbf{x}_t \cdot y_{t-d}^2$ and $\mathbf{x}_t \cdot y_{t-d}^3$, (see Teräsvirta (1998) p.516).

One can thus conclude that a higher-order Taylor series expansion of the transition function is needed in order to devise a linearity test that is free from the deficiencies illustrated above. Now, consider the second-order Taylor series expansion of the transition function $F_t^*(\cdot)$ around the point $\gamma_1 = \gamma_2 = 0$.

$$\begin{aligned} T_2 &= F_t^*(\cdot)|_{\gamma_1=\gamma_2=0} + \frac{\partial F_t^*(\cdot)}{\partial \gamma_1}|_{\gamma_1=\gamma_2=0} * \gamma_1 + \frac{\partial F_t^*(\cdot)}{\partial \gamma_2}|_{\gamma_1=\gamma_2=0} * \gamma_2 \\ &+\frac{1}{2}\frac{\partial^2 F_t^*(\cdot)}{\partial \gamma_1 \partial \gamma_1}|_{\gamma_1=\gamma_2=0} * \gamma_1^2 + \frac{\partial^2 F_t^*(\cdot)}{\partial \gamma_1 \partial \gamma_2}|_{\gamma_1=\gamma_2=0} * \gamma_1\gamma_2 \\ &+\frac{1}{2}\frac{\partial^2 F_t^*(\cdot)}{\partial \gamma_2 \partial \gamma_2}|_{\gamma_1=\gamma_2=0} * \gamma_2^2 + R_1(\gamma_1, \gamma_2, c_1, c_2; y_{t-d}) \\ &= (-\frac{1}{9} * y_{t-d} + \frac{1}{9} * c_1) * \gamma_1 + (\frac{1}{9} * y_{t-d} - \frac{1}{9} * c_2) * \gamma_2 \\ &+(\frac{1}{18} * (y_{t-d} - c_1)^2 + \frac{1}{3} * (\frac{1}{9} * y_{t-d} - \frac{1}{9} * c_1) * (-y_{t-d} + c_1)) * \gamma_1^2 + \\ &+(\frac{1}{3} * (-\frac{1}{9} * y_{t-d} + \frac{1}{9} * c_2) * (-y_{t-d} + c_1) + \frac{1}{3} * (\frac{1}{9} * y_{t-d} - \frac{1}{9} * c_1) * (y_{t-d} - c_2)) * \gamma_2 * \gamma_1 \\ &+(\frac{1}{18} * (y_{t-d} - c_2)^2 + \frac{1}{3} * (-\frac{1}{9} * y_{t-d} + \frac{1}{9} * c_2) * (y_{t-d} - c_2)) * \gamma_2^2 + R_1(\gamma_1, \gamma_2, c_1, c_2; y_{t-d}). \end{aligned}$$

Substitution of the Taylor series expansion expression for the transition function in (11) yields the following auxiliary regression model

$$y_t = \boldsymbol{\beta}_1'\mathbf{x}_t + \boldsymbol{\beta}_2'\mathbf{x}_t \cdot y_{t-d} + \boldsymbol{\beta}_3'\mathbf{x}_t \cdot y_{t-d}^2 + e_t, \tag{14}$$

where $e_t = \boldsymbol{\theta}'\mathbf{x}_t R_2 + u_t$ and

$$\begin{aligned} \boldsymbol{\beta}_1 &= \boldsymbol{\phi} + \frac{1}{9}\gamma_1 c_1 \boldsymbol{\theta} - \frac{1}{9}\gamma_2 c_2 \boldsymbol{\theta} + \frac{1}{54}\gamma_1^2 c_1^2 \boldsymbol{\theta} + \frac{2}{27}\gamma_2\gamma_1 c_2 c_1 \boldsymbol{\theta} + \frac{1}{54}\gamma_2^2 c_2^2 \boldsymbol{\theta} \\ \boldsymbol{\beta}_2 &= \frac{1}{9}\boldsymbol{\theta}(\gamma_2 - \gamma_1) - \frac{1}{27}\gamma_1^2 c_1 \boldsymbol{\theta} - \frac{2}{27}\gamma_2\gamma_1 c_1 \boldsymbol{\theta} - \frac{2}{27}\gamma_2\gamma_1 c_2 \boldsymbol{\theta} - \frac{1}{27}\gamma_2^2 c_2 \boldsymbol{\theta} \\ \boldsymbol{\beta}_3 &= \frac{1}{54}\gamma_1^2 \boldsymbol{\theta} + \frac{2}{27}\gamma_2\gamma_1 \boldsymbol{\theta} + \frac{1}{54}\gamma_2^2 \boldsymbol{\theta}. \end{aligned}$$

The null hypothesis of linearity $\gamma_2 = \gamma_1 = 0$ implies that the parameter vectors $\boldsymbol{\beta}_2$ and $\boldsymbol{\beta}_3$ are equal to zero for the case when $p < d$. The test statistic has a standard χ^2 distribution with $2(p+1)$ degrees of freedom.

When it is the case that $1 \leq d \leq p$ the auxiliary test regression is reported as in equation (15) below.

$$y_t = \beta_{1,0} + \widetilde{\boldsymbol{\beta}}_1'\widetilde{\mathbf{x}}_t + \beta_{2,0} y_{t-d} + \widetilde{\boldsymbol{\beta}}_2'\widetilde{\mathbf{x}}_t y_{t-d} + \beta_{3,0} y_{t-d}^2 + \widetilde{\boldsymbol{\beta}}_3'\widetilde{\mathbf{x}}_t y_{t-d}^2 + e_t$$

where $\widetilde{\boldsymbol{\beta}}_j = \{\beta_{j,1}, ..., \beta_{j,p}\}'$ for $j = 1, 2, 3$.

$$y_t = \beta_{1,0} + \widetilde{\widetilde{\boldsymbol{\beta}}}_1'\widetilde{\mathbf{x}}_t + \widetilde{\widetilde{\boldsymbol{\beta}}}_2'\widetilde{\mathbf{x}}_t y_{t-d} + \widetilde{\boldsymbol{\beta}}_3'\widetilde{\mathbf{x}}_t y_{t-d}^2 + e_t, \tag{15}$$

where

$$\widetilde{\widetilde{\beta}}_{1,i}=\left\{\begin{array}{cc}\widetilde{\beta}_{1,i} & if\ \ i\neq d\\ \widetilde{\beta}_{1,i}+\beta_{2,0} & if\ \ i=d\end{array}\right\}\ i=1,...,p$$

and

$$\widetilde{\widetilde{\beta}}_{2,i}=\left\{\begin{array}{cc}\widetilde{\beta}_{2,i} & if\ \ i\neq d\\ \widetilde{\beta}_{2,i}+\beta_{3,0} & if\ \ i=d\end{array}\right\} i=1,...,p.$$

Hence, when the transition variable y_{t-d} is an element of the vector $\widetilde{\mathbf{x}}_t=\{y_{t-1},...,y_{t-p}\}'$ the linearity test against the BSTAR model with different slope coefficients reduces to testing for the joint insignificance of the auxiliary regression parameters $\widetilde{\widetilde{\boldsymbol{\beta}}}_2$ and $\widetilde{\boldsymbol{\beta}}_3$. The corresponding test statistics have the standard χ^2 distribution with $2p$ degrees of freedom under the null hypothesis.

The linearity test based on test the regression (15) also has power in a situation when the intercept is the only parameter that takes different values in different regimes. This is observed since the parameter vector $\widetilde{\widetilde{\boldsymbol{\beta}}}_2$ only equals zero when all the elements of the parameter vector $\boldsymbol{\theta}$ are equal to zero too, i.e. when $\theta_0=\theta_1=\theta_2$.

Again, notice that the test regression (15) is exactly the same as the test regressions used to carry out the linearity tests against the ESTAR and LSTR2 models, see equation (3.16) in Teräsvirta (1994a) and Teräsvirta (1998) (p.516).

In the case when both slopes parameters in the transition function are equal, the null hypothesis of linearity is

$$H_0:\gamma=0.$$

Then the transition function is

$$\begin{aligned}F_t\left(\gamma,c_1,c_2;y_{t-d}\right) &= \frac{\exp\left[-\gamma\left(y_{t-d}-c_1\right)\right]+\exp\left[\gamma\left(y_{t-d}-c_2\right)\right]}{1+\exp\left[-\gamma\left(y_{t-d}-c_1\right)\right]+\exp\left[\gamma\left(y_{t-d}-c_2\right)\right]}\\ \gamma &> 0,\ c_1>c_2.\end{aligned}$$

The linearity test can be similarly derived using a second-order Taylor series expansion of the BSTAR transition function around the point $\gamma=0:$

$$T_2=F_t\left(\cdot\right)|_{\gamma=0}+\frac{\partial F_t\left(\cdot\right)}{\partial\gamma}|_{\gamma=0}*\gamma+\frac{\partial^2F_t\left(\cdot\right)}{\partial^2\gamma}|_{\gamma=0}*\gamma^2+R_2\left(\gamma,c_1,c_2;y_{t-d}\right).$$

Substituting it into the BSTAR model and rearranging terms results in auxiliary regression (16) below when the value of the delay parameter d is greater than the order of the BSTAR model.

$$y_t=\boldsymbol{\beta}_1'\mathbf{x}_t+\boldsymbol{\beta}_2'\mathbf{x}_t\cdot y_{t-d}+\boldsymbol{\beta}_3'\mathbf{x}_t\cdot y_{t-d}^2+e_t, \tag{16}$$

where $e_t = \boldsymbol{\theta}'\mathbf{x}_t R_2 + u_t$ and $\boldsymbol{\beta}_j = \{\beta_{j,0}, \beta_{j,1}, ..., \beta_{j,p}\}'$ for $j = 1, 2, 3$

$$\begin{aligned}
\boldsymbol{\beta}_1 &= \boldsymbol{\phi} + \frac{1}{9}\gamma\boldsymbol{\theta}\left(c_1 - c_2\right) + \frac{1}{54}\gamma^2\boldsymbol{\theta}\left(c_1^2 + c_2^2\right) + \frac{2}{27}\gamma^2\boldsymbol{\theta}c_1c_2 \\
\boldsymbol{\beta}_2 &= -\frac{1}{9}\gamma^2\boldsymbol{\theta}\left(c_2 - c_1\right) \\
\boldsymbol{\beta}_3 &= \frac{1}{9}\gamma^2\boldsymbol{\theta}.
\end{aligned}$$

Observe that, the null hypothesis of linearity $\gamma = 0$ implies that the regression coefficient vectors $\boldsymbol{\beta}_2$ and $\boldsymbol{\beta}_3$ are equal to zero when the transition variable is not an element of the vector $\widetilde{\mathbf{x}}_t = \{y_{t-1}, ..., y_{t-p}\}'$.

In case the transition variable is an element of the vector $\widetilde{\mathbf{x}}_t = \{y_{t-1}, ..., y_{t-p}\}'$ the corresponding auxiliary test regression is given in (17). To see this, we need to collect similar terms in equation (16), as follows:

$$y_t = \beta_{1,0} + \widetilde{\boldsymbol{\beta}}_1'\widetilde{\mathbf{x}}_t + \beta_{2,0}y_{t-d} + \widetilde{\boldsymbol{\beta}}_2'\widetilde{\mathbf{x}}_t \cdot y_{t-d} + \beta_{3,0}y_{t-d}^2 + \widetilde{\boldsymbol{\beta}}_3'\widetilde{\mathbf{x}}_t \cdot y_{t-d}^2 + e_t$$

and $\widetilde{\boldsymbol{\beta}}_j = \{\beta_{j,1}, ..., \beta_{j,p}\}'$ for $j = 1, 2, 3$.

$$y_t = \beta_{1,0} + \widetilde{\widetilde{\boldsymbol{\beta}}}_1'\widetilde{\mathbf{x}}_t + \widetilde{\widetilde{\boldsymbol{\beta}}}_2'\widetilde{\mathbf{x}}_t \cdot y_{t-d} + \widetilde{\boldsymbol{\beta}}_3'\widetilde{\mathbf{x}}_t \cdot y_{t-d}^2 + e_t, \tag{17}$$

where

$$\widetilde{\widetilde{\beta}}_{1,i} = \left\{ \begin{array}{cc} \widetilde{\beta}_{1,i} & if \ \ i \neq d \\ \widetilde{\beta}_{1,i} + \beta_{2,0} & if \ \ i = d \end{array} \right\} \ i = 1, ..., p$$

and

$$\widetilde{\widetilde{\beta}}_{2,i} = \left\{ \begin{array}{cc} \widetilde{\beta}_{2,i} & if \ \ i \neq d \\ \widetilde{\beta}_{2,i} + \beta_{3,0} & if \ \ i = d \end{array} \right\} i = 1, ..., p.$$

Next, note that the null hypothesis of linearity $\gamma = 0$ implies that the coefficients $\widetilde{\widetilde{\boldsymbol{\beta}}}_2$ and $\widetilde{\boldsymbol{\beta}}_3$ in equation (17) are equal to zero. Hence, the linearity test against the BSTAR model with equal slope coefficients reduces to testing for the joint significance of the auxiliary regression parameters $\widetilde{\widetilde{\boldsymbol{\beta}}}_2$ and $\widetilde{\boldsymbol{\beta}}_3$. The test statistic has a standard χ^2 distribution with $2p$ degrees of freedom under the null hypothesis.

In small samples the asymptotic χ^2 distribution might be a rather poor approximation to the distribution of the test statistics. Therefore Teräsvirta (1998) suggests using the $F-$test instead, which is carried out in the following three steps:

1. calculate the restricted residual sum of squares $SSR_0 = \sum_{t=1}^{T}\left(\widehat{u}_t\right)^2$ after regressing y_t on $\mathbf{x}_t$;

2. calculate the unrestricted residual sum of squares $SSR_1 = \sum_{t=1}^{T} (\widehat{e}_t)^2$ after regressing y_t on $\mathbf{x}_t$, $\mathbf{x}_t \cdot y_{t-d}$, and $\mathbf{x}_t \cdot y_{t-d}^2$ in the case when $p < d$, and y_t on $\mathbf{x}_t$, $\widetilde{\mathbf{x}}_t \cdot y_{t-d}$, and $\widetilde{\mathbf{x}}_t \cdot y_{t-d}^2$ in the case when $1 \leq d \leq p$.

3. calculate

$$F = \frac{(SSR_0 - SSR_1)/df1}{SSR_1/df2} \quad ,$$

where $df1 = 2(p+1)$ and $df2 = T - 3p - 3$ in the case when $p < d$, and $df1 = 2p$ and $df2 = T - 3p - 1$ in the case when $1 \leq d \leq p$. Under the null hypotheses $H_0 : \boldsymbol{\beta}_2 = \boldsymbol{\beta}_3 = 0$ and $H_0 : \widetilde{\widetilde{\boldsymbol{\beta}}}_2 = \widetilde{\boldsymbol{\beta}}_3 = 0$ the respective test statistics have approximate distributions $F(2(p+1), T - 3p - 3)$ and $F(2p, T - 3p - 1)$.

6 Misspecification testing.

The misspecification tests that check the adequacy of the estimated STAR family models comprise an important part of the modelling cycle. Of interest are the test of no autocorrelation, the test of parameter constancy and the test of no remaining nonlinearity. All these tests were first suggested in Eirtheim and Teräsvirta (1996) for the LSTR1 and ESTAR models. In this paper they are adapted to the BSTAR models either with or without the restriction of equal slope parameters. As all details of the tests can be found in the reference above, I will only briefly outline what should be changed in order to adapt them to the BSTAR model.

In fact, only the expressions for the first partial derivatives of the BSTAR transition function with respect to the parameters of the respective transition functions need to be derived. First, consider the BSTAR model without imposing the restriction of equal slope parameters γ_1 and γ_2. The needed partial derivatives are

$$\begin{aligned}
\frac{\partial F_t}{\partial \gamma_1} &= -(\boldsymbol{\theta}'\mathbf{x}_t) \frac{(y_{t-d} - c_1)}{sc} \exp\left(-\gamma_1 \frac{[y_{t-d} - c_1]}{sc}\right) [1 - F_t(y_{t-d})]^2 \\
\frac{\partial F_t}{\partial c_1} &= (\boldsymbol{\theta}'\mathbf{x}_t) \frac{\gamma_1}{sc} \exp\left(-\gamma_1 \frac{[y_{t-d} - c_1]}{sc}\right) [1 - F_t(y_{t-d})]^2 \\
\frac{\partial F_t}{\partial \gamma_2} &= (\boldsymbol{\theta}'\mathbf{x}_t) \frac{(y_{t-d} - c_2)}{sc} \exp\left(\gamma_2 \frac{[y_{t-d} - c_2]}{sc}\right) [1 - F_t(y_{t-d})]^2 \\
\frac{\partial F_t}{\partial c_2} &= -(\boldsymbol{\theta}'\mathbf{x}_t) \frac{\gamma_2}{sc} \exp\left(\gamma_2 \frac{[y_{t-d} - c_2]}{sc}\right) [1 - F_t(y_{t-d})]^2 .
\end{aligned}$$

Second, consider the BSTAR model with the equal slope parameter restriction imposed. In this case, the vector of the first derivatives of the corresponding

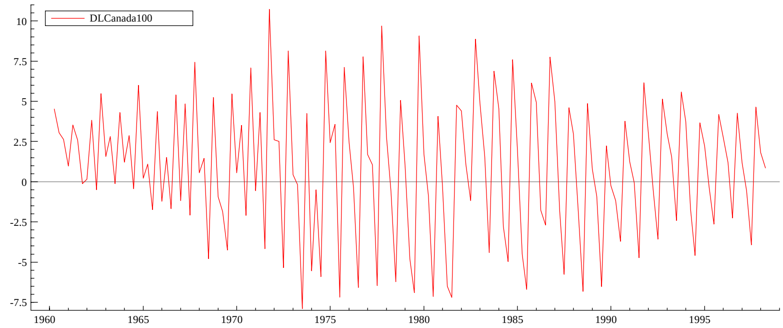

Figure 12: IIP Canada, growth rate, $\times 10^2$.

transition function reads:

$$\begin{aligned}
\frac{\partial F_t}{\partial \gamma_1} &= -\left(\boldsymbol{\theta}'\mathbf{x}_t\right)\frac{(y_{t-d}-c_1)}{sc}\exp\left(-\gamma_1\frac{[y_{t-d}-c_1]}{sc}\right)\left[1-F_t\left(y_{t-d}\right)\right]^2 \\
&\quad+\left(\boldsymbol{\theta}'\mathbf{x}_t\right)\frac{(y_{t-d}-c_2)}{sc}\exp\left(\gamma_1\frac{[y_{t-d}-c_2]}{sc}\right)\left[1-F_t\left(y_{t-d}\right)\right]^2 \\
\frac{\partial F_t}{\partial c_1} &= \left(\boldsymbol{\theta}'\mathbf{x}_t\right)\frac{\gamma_1}{sc}\exp\left(-\gamma_1\frac{[y_{t-d}-c_1]}{sc}\right)\left[1-F_t\left(y_{t-d}\right)\right]^2 \\
\frac{\partial F_t}{\partial c_2} &= -\left(\boldsymbol{\theta}'\mathbf{x}_t\right)\frac{\gamma_1}{sc}\exp\left(\gamma_1\frac{[y_{t-d}-c_2]}{sc}\right)\left[1-F_t\left(y_{t-d}\right)\right]^2.
\end{aligned}$$

7 Empirical example.

As an illustration of the empirical relevance of the suggested model, the modeling strategy suggested in Teräsvirta (1998) is applied to the industrial production index of Canada[3] taken from the OECD database. The seasonally unadjusted quarterly data span the period from 1960:1 until 1998:2. The sample size is 154 observations.

The first difference of the logarithmic transformation of the time series is displayed in Figure 12. Note that in order to avoid having to deal with small numbers the transformed time series was multiplied by 100.

In the first step a linear autoregressive model is fitted to the data at hand:

$$\begin{aligned}
y_t &= \underset{(0.080)}{0.307}\,y_{t-1}+\underset{(0.082)}{0.127}\,y_{t-2}-\underset{(0.079)}{0.031}\,y_{t-3}+\underset{(0.075)}{0.327}\,y_{t-4} \\
&\quad-\underset{(0.072)}{0.289}\,y_{t-5}-\underset{(0.076)}{0.170}\,y_{t-6}-\underset{(0.077)}{0.071}\,y_{t-7}+\underset{(0.074)}{0.245}\,y_{t-8} \\
&\quad+\underset{(0.622)}{0.429}\,S_1\underset{(0.584)}{-0.207}\,S_2\underset{(0.580)}{-1.284}\,S_3\underset{(0.583)}{+2.956}\,S_4
\end{aligned}$$

[3] This data were kindly provided by Birgit Strikholm.

Table 1: Linearity test.

	F-test	p-value	df1	df2
$a)^1$	1.0477	0.4105	12	121
$b)^2$	1.9540	0.0106	24	109

[1] based on a first-order of Taylor series expansion of the transition function.
[2] based on a second-order of Taylor series expansion of the transition function.

$T = 145$, $R^2 = 0.847$, $\widehat{\sigma}_{OLS_{AR(8)}} = 1.671$, $F_{AR_{1-5}}(5, 128) = 0.675[0.643]$,
$F_{ARCH4}(4, 125) = 0.537[0.708]$, $\chi^2_{normality}(2) = 4.672[0.097]$,
$F_{x_i^2}(19, 113) = 0.911[0.571], F_{RESET}(1, 132) = 0.651[0.421]$.

Under the estimated regression the following useful information is provided: $T-$ a sample size; R^2- a squared multiple correlation coefficient; $\widehat{\sigma}_{OLS_{AR(8)}}-$ a standard deviation of residuals; $F_{AR_{1-5}}-$ a F-test for 5^{th} order of residual autocorrelation, Godfrey (1978); $F_{ARCH4}-$ a F-test for 4^{th} order of ARCH, Engle (1982); $\chi^2_{normality}-$ a chi-square test for normally distributed residuals, Doornik and Hansen (1994); $F_{x_i^2}$ $-$ a F-test for heteroscedasticity using squares of regressors, White (1980); $F_{RESET}-$ a F-test for functional form mis-specification, Ramsey (1969). All the tests were calculated using PcGive 9.1, see Hendry and Doornik (1999). Notice that the standard errors are reported in parentheses under the parameter estimates.

The $AR(8)$ model above was used as a basis for the linearity test as it seems to remove autocorrelation in residuals and satisfy the other model design criteria. The results of the test using the trend as a transition variable, are shown in Table 1. The test statistic was calculated using first- and second-order Taylor series expansions of the BSTAR transition function.

Note that the linearity test based on the first-order Taylor expansion accepts the null of linearity. However, the test based on the second-order expansion rejects the null hypothesis of linearity at the 5% significance level. Hence, the next step is nonlinear modeling of the time series in question. The estimated non-linear model has the general form:

$$y_t = \boldsymbol{\phi}'\widetilde{\mathbf{x}}_t + \mathbf{S}^{\phi\prime}\mathbf{D}_t + \left[\boldsymbol{\theta}'\widetilde{\mathbf{x}}_t + \mathbf{S}^{\theta\prime}\mathbf{D}_t\right] F_t\left(t/T\right) + u_t \quad ,$$

where $\widetilde{\mathbf{x}}_t = (y_{t-1}, ..., y_{t-p})'$ and $\mathbf{D}_t$ is a vector of the seasonal dummies. $\mathbf{S}^{\phi} = (S_1^{\boldsymbol{\phi}}, ..., S_4^{\boldsymbol{\phi}})'$ and $\mathbf{S}^{\boldsymbol{\theta}} = (S_1^{\boldsymbol{\theta}}, ..., S_4^{\boldsymbol{\theta}})'$ are seasonal parameter vectors, whereas $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$ are autoregressive parameter vectors. $F_t\left(t/T\right)$ is the BSTAR transition function (6), where the trend variable t, divided by the total number of available observations T, is used as the transition variable. Hence, the transition variable is normalized to take values in the interval $(0, 1]$.

First, the BSTAR model was estimated using a maximum of eight lags of the dependent variable both in the linear as well as nonlinear parts of the model.

The estimation results (not reported in order to save the space) were unsatisfactory as the various model specification assumptions were violated. The model that passes all the misspecification tests is given in (18) below. Observe that together with the usual exclusion restrictions $\phi_i = 0$ and $\theta_i = 0$, restrictions of the type $\varphi_i = -\theta_i$ were imposed on the coefficients that correspond to $9th$, $10th$, and $11th$ lags as well as on the coefficient that corresponds to the first seasonal dummy S_1.

$$\begin{aligned} y_t = \; & \underset{(0.055)}{0.397} y_{t-1} + \underset{(0.064)}{0.274} y_{t-2} - \underset{(0.118)}{0.262} y_{t-4} - \underset{(0.066)}{0.185} y_{t-5} \\ & - \underset{(0.113)}{0.141} y_{t-9} + \underset{(0.105)}{0.304} y_{t-10} - \underset{(0.107)}{0.338} y_{t-11} \\ & + \underset{(1.342)}{6.372} S_1 - \underset{(1.429)}{8.890} S_2 - \underset{(1.144)}{5.810} S_3 + \underset{(1.33)}{10.620} S_4 + \\ & [- \underset{(0.086)}{0.261} y_{t-3} + \underset{(0.150)}{0.551} y_{t-4} + \underset{(0.113)}{0.141} y_{t-9} \\ & - \underset{(0.105)}{0.304} y_{t-10} + \underset{(0.107)}{0.338} y_{t-11} + \underset{(0.075)}{0.330} y_{t-16} + \\ & - \underset{(1.342)}{6.372} S_1 + \underset{(1.361)}{6.942} S_2 + \underset{(1.306)}{4.967} S_3 - \underset{(1.375)}{7.561} S_4] * \widehat{F}_t (t/T), \end{aligned} \tag{18}$$

$$\widehat{F}_t (t/T) = \frac{\exp\left(-\frac{\underset{(2.749)}{7.129}}{\widehat{\sigma}_{t/T}} \left[t/T - \underset{(0.019)}{0.339} \right] \right) + \exp\left(\frac{\underset{()}{652.06}}{\widehat{\sigma}_{t/T}} \left[t/T - \underset{(0.00022)}{0.707} \right] \right)}{1 + \exp\left(-\frac{\underset{(2.749)}{7.129}}{\widehat{\sigma}_{t/T}} \left[t/T - \underset{(0.019)}{0.339} \right] \right) + \exp\left(\frac{\underset{()}{652.06}}{\widehat{\sigma}_{t/T}} \left[t/T - \underset{(0.00022)}{0.707} \right] \right)},$$

$T = 137$, $\widehat{\sigma}_{BSTAR} = 1.355$, $\widehat{\sigma}_{t/T} = 0.289$, $\chi^2_{ARCH4}(4) = 2.816\,[0.588]$, $\chi^2_{normality}(2) = 2.335[0.311]$, $\widehat{\sigma}_{BSTAR}/\widehat{\sigma}_{OLS_{AR(16)}} = 1.355/1.597 = 0.85$.

Standard errors are reported in parentheses under the parameter estimates. Also notice that the standard error is not reported for the second threshold parameter estimate $\widehat{\gamma}_2$. This is because the estimated slope parameter is so steep that for the given sample size the number of observations that lie in the very neighborhood of the corresponding location parameter estimate $\widehat{c}_2$ is not sufficient to ensure the full rank of the reported information marix, see Figure 14 and Section 4 for an explanation of the problem.

The estimated residual values, correlogram, and density are reported in Figure 13. Actual and fitted values along with the estimated transition function are depicted in Figure 14. The ratio of standard deviations of the residuals of the BSTAR and linear AR(16) models is $\widehat{\sigma}_{BSTAR}/\widehat{\sigma}_{OLS_{AR(16)}} = 1.355/1.597 = 0.85$, such that the variance of the BSTAR residuals is only about 72% of that of the linear AR(16) model. Figure 15 displays estimated residuals from the nonlinear BSTAR and linear AR(16) models.

Table 2: LM test of no serial correlation.

$F-test$	$p-value$	$df1$	$df2$	$F-test$	$p-value$	$df1$	$df2$
0.011	0.916	1	114	1.229	0.289	8	100
0.047	0.953	2	112	1.111	0.362	9	98
0.525	0.665	3	110	0.985	0.461	10	96
0.750	0.559	4	108	0.871	0.570	11	94
0.794	0.556	5	106	0.821	0.628	12	92
0.724	0.630	6	104	1.063	0.400	13	90
0.649	0.713	7	102				

Table 3: LM test of parameter constancy.

	F-test	p-value	df1	df2
All	0.811	0.780	51	65
All linear	1.179	0.286	21	95
All nonlinear	0.754	0.806	30	86
AR linear	0.756	0.693	12	104
Dummies linear	0.593	0.800	9	107
AR nonlinear	0.878	0.605	18	98
Dummies nonlinear	0.571	0.860	12	104

Table 4: LM test of no remaining nonlinearity.

TV[1]	F-test	p-value	df1	df2	TV	F-test	p-value	df1	df2
y_{t-1}	1.421	0.090	48	68	y_{t-10}	0.923	0.611	48	68
y_{t-2}	1.340	0.131	48	68	y_{t-11}	1.264	0.185	48	68
y_{t-3}	0.959	0.555	48	68	y_{t-12}	0.668	0.929	48	68
y_{t-4}	0.820	0.763	48	68	y_{t-13}	0.716	0.888	48	68
y_{t-5}	1.427	0.087	48	68	y_{t-14}	0.728	0.876	48	68
y_{t-6}	1.134	0.312	48	68	y_{t-15}	0.777	0.819	48	68
y_{t-7}	1.068	0.395	48	68	y_{t-16}	0.803	0.786	48	68
y_{t-8}	0.714	0.890	48	68	trend	1.235	0.213	60	68
y_{t-9}	0.982	0.520	48	68					

[1] TV - transition variable.

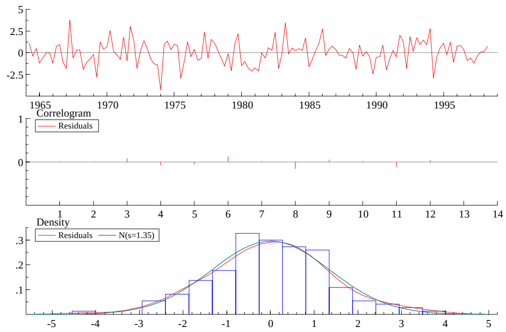

Figure 13: IIP Canada, growth rate. BSTAR model (18). Residuals, its correlogram, histogram, and estimated density.

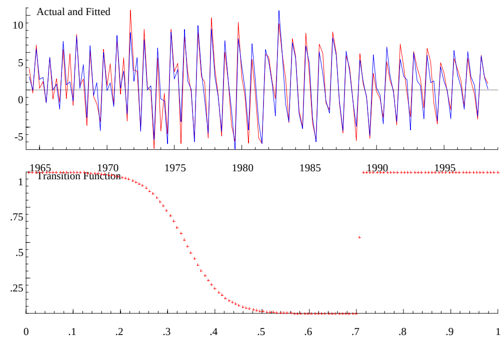

Figure 14: IIP Canada, growth rate. Actual and fitted values of a BSTAR model (18). Transition function.

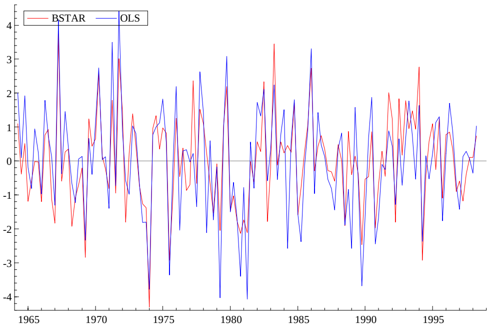

Figure 15: IIP Canada, growth rate. Estimated residuals of a BSTAR model (18) and a linear AR(16) model.

8 Conclusion.

The present paper has introduced a nonlinear model that is related to the STAR class of models. The so-called BSTAR model, which here is fitted to the growth rate of the Canadian Index of Industrial Production time series, is a generalization of the three regime LSTR2 model suggested earlier in Teräsvirta (1998). The LSTR2 model has one slope parameter and two threshold parameters, such that the speed of transition between the outer and the middle regimes - measured by the slope parameter - is the same. In contrast, the suggested BSTAR model possesses two slope and two threshold parameters, such that when the slope parameters differ from each other, different speeds of transition between the outer-lower and the middle regimes, as well as between the middle regime and the outer-upper regimes, is allowed for. In the case with equal slope parameters, the BSTAR model closely substitutes the LSTR2 model.

In addition, the fact that the three-regime BSTAR model has two generally different threshold parameters helps avoid the unfortunate properties of the ESTAR and AESTAR models in a situation when the only slope parameter in these models is rather large.

References

Anderson, H. M. (1997): “Transaction Costs and Non-Linear Adjustment Towards Equilibrium in the US Treasury Bill Market,” *Oxford Bulletin of Economics and Statistics*, 59(4), 465–484.

Doornik, J. A. (1999): *Object-Oriented Matrix Programming Using Ox Version 2.1*. London: Timberlake Consultants Press, 3rd edn.

Doornik, J. A., and H. Hansen (1994): “A Practical Test for Univariate and Multivariate Normality,” Discussion Paper, Nuffield College, Oxford.

Eirtheim, Ø., and T. Teräsvirta (1996): "Testing the Adequacy of Smooth Transition Autoregressive Models," *Journal of Econometrics*, 74, 56–75.

Engle, R. F. (1982): "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation," *Econometrica*, 50, 987–1007.

Godfrey, L. G. (1978): "Testing for Higher Order Serial Correlation in Regression Equations When the Regressors Include Lagged Dependent Variables," *Econometrica*, 46, 1303–1313.

Granger, C. W. J., and T. Terasvirta (1993): *Modeling Nonlinear Economic Relationships*, Advanced Texts in Econometrics. Oxford University Press.

Hendry, D. F., and J. A. Doornik (1999): *Empirical Econometric Modelling Using PcGive 9.0 for Windows*. London: Timberlake Consultants Press, 2nd edn.

Luukkonen, R., P. Saikkonen, and T. Terasvirta (1988): "Testing Linearity Against Smooth Transition Autoregressive Models," *Biometrika*, 75, 491–499.

Ramsey, J. B. (1969): "Tests for Specification Errors in Classical Linear Leat-Squares Regression Analysis," *Journal of the Royal Statisitical Society, Series B*, 31, 350–371.

Saikkonen, P., and R. Luukkonen (1988): "Lagrange Multiplier Tests for Testing Non-Linearities in Time Series Models," *Scandinavian Journal of Statistics*, 15, 55–68.

Teräsvirta, T. (1994a): "Specification, Estimation and Evaluation of Smooth Transition Autoregressive Models," *Journal of American Statistical Association*, 89(425), 208–218.

——— (1994b): "Testing Linearity and Modelling Nonlinear Time Series," *Kybernetika*, 30, 319–330.

——— (1998): "Modeling Economic Relationships with Smooth Transition Regressions," in *Handbook of Applied Economic Statistics*, ed. by A. Ullah, and D. E. A. Giles, chap. 15, pp. 507–552. Marcel Dekker, Inc.

Tong, H. (1990): *Non-Linear Time Series: A Dynamic System Approach*, Oxford Statistical Science Series. Clarendon Press Oxford.

van Dijk, D. (1999): "Smooth Transition Models: Extensions and Outlier Robust Inference," Ph.D. thesis, Econometric Institute, Erasmus University Rotterdam.

van Dijk, D., T. Teräsvirta, and P. H. Franses (2000): "Smooth Transition Autoregressive Models - a Survey of Recent Developments," SSE/EFI Working Paper Series in Economics and Finance, No. 380.

White, H. (1980): "A Heteroscedastic-Consistent Covariance Matrix Estimator and a Direct Test for Heteroscedasticity," *Econometrica*, 48, 817–838.

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