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Investment under Uncertainty – the Case of Repeated Investment Options

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Abstract

This paper considers optimal investment behaviour when investment options evolve deterministically or stochastically over time and investments are irreversible and indivisible. It extends the standard investment-under-uncertainty set-up with a single investment option to the case of repeated options. Analytical solutions are derived for the deterministic case and for the case of a geometric Brownian motion. It is argued that when investment options are repeated, the simple net-present-value rule in general fares better as an investment criterion than the rule derived from the single-option approach. Furthermore, sensitivity analyses reveal that the effects of parameter changes are very different when using the repeated-options approach instead of the single-option approach.

Keywords: geometric Brownian motion, indivisibility, investment, irreversibility, repeated options, replacement, stochastic processes, technology, uncertainty.

JEL: D1, D9, O3, Q12.

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1 Introduction

During the last two decades a large number of studies have analysed the real-option approach to investment decisions and Dixit and Pindyck (1994) present a unified account of this approach. The object of analysis is optimal investment behaviour when: i) investments are irreversible and indivisible; ii) there is uncertainty about the future, e.g. about the cost and profitability of an investment; and iii) there is an option to postpone the investment. It is argued that incorporating these features into the optimisation problem often provides a more appropriate description of the investment decision than more orthodox and static formulations using a simple net-present-value (NPV) criterion. Furthermore, since i) - iii) usually create a (considerable) value of waiting, the derived decision rules are often significantly different from those implied by the simple NPV criterion.

The core model in Dixit and Pindyck (1994), and in much of the related literature, is a single-option model first developed by McDonald and Siegel (1986). In this model, the option is "killed" when the investment is undertaken. Hence no future re-investment can take place. While this type of model has rightly received considerable attention in the literature, there exist situations where it is inappropriate.

As an example, think of an individual or a firm considering when to invest in new IT equipment. New and more productive IT equipment is continuously introduced at the market, and prices are constantly changing. Hence the option value aspect is clearly present in this decision problem. However, the agent knows that the current IT investment is not a unique event. In a few years, it will be optimal for him to buy an even bigger IT system. Therefore, being rational, he cannot consider the current investment option as independent of future options. When he buys new IT equipment, he "kills" the current option, but he immediately receives a new option – the option to buy even newer IT equipment. The optimal timing of his current investment is therefore likely to depend on – and influence – the pattern of future investments. Hence, these considerations should be incorporated explicitly into a formulation of the decision problem of the agent.

The aim of the present paper is to extend the methodology of the real-option literature to include these aspects, and to analyse the implications for optimal investment behaviour. Central questions are: Do the insights derived from the single-option model carry over to the case of repeated options? Or does the single-option model yield invalid descriptions of optimal behaviour when investment options are actually repeated? Furthermore, are the effects of changes in the underlying parameters different in the two models?

More specifically, the intertemporal decision problem is modelled as follows: In each period, production is assumed to depend only on the productivity of installed technology. There is no effort or input of other factors. However, productivity may be increased by investing in the currently available technology in the economy. The productivity of this exogenous technology is assumed to evolve according to a stochastic process. All investments are irreversible and indivisible, in the sense that the agent has to pay for the productivity *level* of the exogenous technology, not just the *difference* in productivity levels between installed and exogenous technology, which would make continuous (or incremental) investments optimal.

An analytical solution is derived for the case where the productivity of exogenous technology evolves according to a geometric Brownian motion. The derived optimal decision rule is compared to those derived from the simple net present value (NPV) criterion and the single-option criterion of Dixit and Pindyck (1994). An important finding is that when investment options are repeated as in this set-up, the simple NPV rule is often a better guide to optimal investments than the rule derived from the single-option approach. In addition, the implications of changes in growth rates, interest rates, and uncertainty levels for optimal investment behaviour under repeated options are very different from – and often opposite to – those found using the single-option approach. Instead, the effects resemble those of the simple NPV approach.

The problem analysed in this paper can also be interpreted as an optimal replacement problem. When an investment is made, the installed technology is replaced by a new and more productive technology. Replacement problems of this sort have been subject to much research in the literature. However, only few previous studies have similarities to this study. Ye (1990) analyses a replacement problem where maintenance and operation cost of installed equipment follows an Ito process. By investing in new equipment, the process is returned to its initial state. With everything else constant, Ye (1990) derives the closed form decision rule for the case of a simple Brownian motion. More recently, Mauer and Ott (1995) analyse a related replacement problem, where maintenance and operation cost follows a geometric Brownian motion. As in Ye (1990), the process is reset to its initial value whenever an investment in new equipment is made.

In both these papers, the uncertainty is related to the installed technology. In the present paper, the problem is turned upside down. Instead of focusing on operation and maintenance cost as the variable of interest, uncertainty is introduced into the exogenous investment options. This change is considered to be of strong empirical relevance. The IT example above is just one example illustrating that with modern technology, operation and maintenance cost is no longer of major importance when evaluating replacement.

It is the productivity of installed technology relative to newer available technology which is at the core of many replacement decisions. However, even though the interpretations are different, there is a close mathematical correspondence between the previous replacement models and the model in this paper. An appendix to this paper shows this more formally, by deriving an analytical solution for the more general replacement problem, where both the productivity of installed and the productivity of exogenous technology evolve according to geometric Brownian motions.

Apart from the differences in interpretation, Ye (1990) does not obtain as general and yet simple analytic solutions as in the present paper. Furthermore, Ye (1990) does not analyse the model from an option pricing perspective and hence does not provide the interpretations given here. Mauer and Ott (1995), on the other hand, impose a restriction on the level of the stochastic process which makes them unable to derive any analytical solutions. Furthermore, by aiming at a comparison of the derived decision rule with that of the single-option approach, the focus of the present paper is entirely different from that of both Ye (1990) and Mauer and Ott (1995).

The rest of the paper is organised as follows: In Section 2, a deterministic version of the model is presented to build intuition. Section 3 presents the setup of the stochastic model, using a Geometric Brownian Motion. The stochastic model is solved and the optimal decision rule is derived in Section 4. Section 5 contains a comparison of the derived decision rule with those of the simple NPV approach and the single-option approach. Section 6 concludes the paper.

2 The Deterministic Model

The model is a continuous-time model, and the agent is assumed to maximise the net present value of all future income streams. Income is derived solely from production, and the only costs are those associated with new investments in technology. Thus, the model abstracts from decisions regarding other factor inputs. This is done in order to keep it analytically tractable and to focus on the dynamics of investment.

Production, y_t, at time t is given by a function of the non-stochastic productivity level, Θ_t, of installed technology:

$$y_t = F(\Theta_t)$$

where $F' > 0$. The agent can improve this productivity level by investing in exogenous technology. By doing this, he obtains the current productivity level, θ_t, of the exogenous technology process. His productivity is then given by θ_t until the next date of investment. Meanwhile, the exogenous productivity level is assumed to grow according to some exogenous process. The cost of investing in the new technology is given by $C(\theta_t)$, where $C' > 0$.

The decision problem of the agent is thus reduced to determining the timing of investments. He can only invest in the currently available productivity level, and he keeps this level until the next date of investment.

For the present, assume that the initial productivity level of installed technology, Θ_0, is the same as the initial productivity level of exogenous technology, θ_0. Wealth, W, defined as the net present value of future income streams at the initial date, t_0, can then be expressed as a function of the investment dates, $t_1, t_2, ..., t_\tau, ...$:

$$W(t_1, t_2, ...) = \sum_{i=1}^{\infty} \left(\int_{t_{i-1}}^{t_i} e^{-rt} F\left(\theta_{t_{i-1}}\right) dt - e^{-rt_i} C(\theta_{t_i}) \right)$$

where r is the exogenous rate of interest. Thus, the agent maximises W with respect to the investment dates. The first order condition for t_τ can be written as:

$$e^{-rt_\tau} F\left(\theta_{t_{\tau-1}}\right) - e^{-rt_\tau} F\left(\theta_{t_\tau}\right) + \int_{t_\tau}^{t_{\tau+1}} e^{-rt} \frac{dF\left(\theta_{t_\tau}\right)}{dt_\tau} dt + re^{-rt_\tau} C(\theta_{t_\tau}) - e^{-rt_\tau} \frac{dC(\theta_{t_\tau})}{dt_\tau} = 0 \quad (1)$$

The first term minus the second term expresses the immediate loss in marginal production from postponing the investment. The third term represents the increase in the marginal product of the new technology obtained from postponing the investment. The last two terms express the ambiguous effect on the cost of the investment from waiting. On the one hand, the present value of the cost will decrease as a consequence of discounting over an extended period of time. This effect is captured by the fourth term. On the other hand, the undiscounted cost of the investment might change when the date of investment, and therefore the technology level, changes. This is the fifth term.

Further assumptions are required to obtain a more precise characterisation of optimal behaviour.[1] Therefore, assume now that the productivity of exogenous technology evolves according to:

$$\theta_t = \theta_0 e^{\alpha t}$$

[1]In the set-up above, it might even be the case that it is optimal never to invest, or perhaps to invest only a finite number of times. It depends on the process of technological productivity, the involved functional forms, and the parameter values.

where $0 < \alpha < r$. Furthermore, let production take the simple form:

$$F(\Theta_t) = \Theta_t$$

and assume that the cost of investment is linear in θ_t:

$$C(\theta_t) = c\theta_t$$

where $c < \frac{1}{r}$.

The parameters a, c, and r thus satisfy: $\infty > \frac{1}{\alpha} > \frac{1}{r} > c > 0$. These assumptions are needed to ensure that the problem is well defined and that investments are profitable. If $r \leq \alpha$, then optimal wealth will become infinite, which is not an interesting case. If $c \geq \frac{1}{r}$, it will never be profitable to undertake an investment. However, with $\infty > \frac{1}{\alpha} > \frac{1}{r} > c$, it cannot be optimal to postpone investments forever. It must be optimal to invest at some finite point in time, and hence to repeat the investment within another finite horizon.

The above specifications also imply that the problem is *stationary*, i.e. optimal decisions and values do not depend on time *per se*. Too see this, note that wealth, W, can now be written as:

$$W(t_1, t_2, ...) = \theta_0 \sum_{i=1}^{\infty} \left(e^{\alpha t_{i-1}} \int_{t_{i-1}}^{t_i} e^{-rt} dt - ce^{(\alpha - r)t_i} \right)$$

It follows that the optimal investment dates must be independent of θ_0. Hence, the decision problem is identical every time an investment has been undertaken – only the productivity level of installed and exogenous technology has changed. This implies that the time span, s, between investments must be constant, or equivalently, since θ is growing exponentially, that an investment will take place whenever $\theta = \lambda\Theta$ for some constant $\lambda > 1$. Furthermore, optimal wealth, V, must be homogenous of degree one in θ_0.

The fact that the time span between investments is constant, $s = t_\tau - t_{\tau-1}$, can be used to rewrite the first order condition, (1), as:

$$e^{-rs} - \frac{r}{\alpha} e^{-\alpha s} = \left(1 - \frac{r}{\alpha}\right)(1 - rc) \tag{2}$$

Equation (2) gives the optimal time span, s, indirectly as a function of r, α, and c.[2] The condition for s can also be recast in terms of λ, using that

[2] To see that (2) gives a unique value of s, note that the left hand side increases monotonically from $1 - \frac{r}{\alpha}$ to 0 as s goes from 0 to ∞. The right hand side is constant in s, strictly larger than $1 - \frac{r}{\alpha}$, and strictly smaller than 0. Thus, there is exactly one positive value of s satisfying (2).

$\lambda = e^{\alpha s}$:

$$\lambda e^{-\frac{r}{\alpha}} - \frac{r}{\alpha}\lambda^{-1} = \left(1 - \frac{r}{\alpha}\right)(1 - rc) \tag{3}$$

A closer interpretation of this condition will be offered in Section 5.

3 The Stochastic Model

The stochastic version of the model resembles the simple version of the deterministic model presented in the previous section. The only change is that the productivity of exogenous technology is now assumed to evolve according to a geometric Brownian motion:

$$d\theta = \alpha\theta dt + \sigma\theta dz$$

where α and σ are the parameters of the process, and dz is the increment of a Wiener process. As in the deterministic case, it is assumed that the parameters a, c, and r satisfy: $\infty > \frac{1}{\alpha} > \frac{1}{r} > c$.

Note that the decision problem of the agent can be interpreted as an "optimal stopping problem". At each point in time, the agent must decide whether to *continue* producing with productivity level Θ_t, or to *stop* and adopt the current productivity level, θ_t, of the exogenous technology process.

As in the deterministic case, the specification implies that the problem is *stationary*, i.e. optimal decisions and values do not depend on time *per se*. All relevant information is captured in the variables θ and Θ, together with the parameters of the problem. Thus, drop time subscripts and let $V(\theta, \Theta)$ be the value function of the agent when the productivity of installed technology is Θ and that of the exogenous technology is θ. The value function is then given by:

$$\begin{aligned} V(\theta, \Theta) = \max \big\{ &\Theta dt + (1 + rdt)^{-1} E\left[V(\theta + d\theta, \Theta) | \theta\right], \\ &\theta dt - c\theta + (1 + rdt)^{-1} E\left[V(\theta + d\theta, \theta) | \theta\right] \big\} \end{aligned} \tag{4}$$

The first argument on the right-hand side is the value of continuing with the given technology for at least one more "short" time interval, dt. This will yield immediate output, Θdt, and result in a change of productivity of exogenous technology to $\theta + d\theta$. This in turn changes the expectation of the value function, which is discounted by $(1 + rdt)^{-1}$. Likewise, the second term is the value of stopping, i.e. switching to the technology producing θ. This immediately costs $c\theta$ and yields current output θdt in the next short interval of time. After that interval, the value function will be $V(\theta + d\theta, \theta)$, since the productivity of exogenous technology changes as above, whereas the productivity of installed technology now equals θ.

Note that the chosen form of the stochastic process implies that the productivity of exogenous technology can get below the productivity of installed technology. This may at first glance seem unreasonable. However, it might reflect that though the technology in itself has improved, the production from the technology might not have. This could be due to lack of knowledge about the new technology or lack of supportive infrastructure. Not all inventions are equally profitable to everyone.

4 Solving the Model

The simple structure of the problem implies that there exists a unique stopping line, $\theta(\Theta)$, which separates the stopping region from the continuation region. For values of θ above this line, it will be optimal to stop (invest), and for values below, it will be optimal to continue with the installed technology. Furthermore, the stationarity of the problem implies that only relative values of θ and Θ are relevant for the optimal decisions. Scaling θ and Θ with the same factor will not change the decision problem of the agent, but will merely scale the optimal wealth by the same factor. Thus, $V(\theta, \Theta)$ is homogenous of degree one in θ and Θ, and the stopping line must be given by $\theta(\Theta) = \lambda\Theta$ for some constant $\lambda > 1$.

Now, for points belonging to the continuation region, it follows from (4) that:

$$rV(\theta, \Theta)\, dt = \Theta dt + E[dV] \tag{5}$$

Expanding the last term on the right-hand side by use of Ito's Lemma, and dividing through by dt, (5) can be rewritten as:

$$rV = \Theta + \theta\alpha V'_\theta + \frac{1}{2}\theta^2\sigma^2 V''_{\theta\theta} \tag{6}$$

where the arguments of the value function have been suppressed. V'_θ and $V''_{\theta\theta}$ are the first and second order partial derivatives with respect to θ.

The expression in (6) is a second order partial differential equation. However, the homogeneity of $V(\theta, \Theta)$ implies that a normalised value function, $v(w)$, can be defined as:

$$v(w) = \Theta^{-1} V(\theta, \Theta)$$

where $w = \theta/\Theta$. The derivatives of $V(\theta, \Theta)$ with respect to θ can then be expressed as:

$$V'_\theta = v' \quad , \quad V''_{\theta\theta} = \Theta^{-1} v''$$

By use of the normalised value function, the partial differential equation in (6) can be rewritten as an ordinary differential equation:

$$rv = 1 + w\alpha v' + \frac{1}{2} w^2 \sigma^2 v'' \tag{7}$$

which has the following general solution:

$$v = A_1 w^{a_1} + A_2 w^{a_2} + K \tag{8}$$

where A_1, A_2, a_1, a_2, and K are constants to be determined. Now, substituting the solution in (8) back into (7) implies that K must equal r^{-1}, and that a_1 and a_2 must be the roots of the following quadratic equation:

$$\frac{1}{2}\sigma^2 a^2 + a\left(\alpha - \frac{1}{2}\sigma^2\right) - r = 0 \tag{9}$$

Hence $a_1 > 1$ and $a_2 < 0$.

Note that zero is an absorbing state for the productivity of exogenous technology, i.e. if θ becomes equal to zero, it remains equal to zero. In this case, it will never be optimal to invest. Hence $V(0, \Theta)$ must equal Θr^{-1}, which implies that $v(0) = r^{-1}$. Thus, A_2 must equal 0, since $a_2 < 0$, and the value function for the continuation region can then be written more compactly as:

$$v = A_1 w^{a_1} + \frac{1}{r} \tag{10}$$

with:

$$a_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2}\right)^2 + \frac{1}{4} + \frac{2r - \alpha}{\sigma^2}} > 1 \tag{11}$$

The constant A_1 has yet to be determined together with the stopping rule, λ. For this purpose, the value-matching and the smooth-pasting conditions are used. The value matching condition:

$$v(\lambda) = -c\lambda + \lambda v(1) \tag{12}$$

says that at the stopping line, where $w = \lambda$, the normalised value of continuation (the left-hand side) should equal the normalised value of stopping (the right-hand side). Furthermore, these values should meet smoothly, which gives a smooth-pasting condition:

$$v'(\lambda) = -c + v(1) \tag{13}$$

Now, using the smooth-pasting condition to substitute for $v(1)$ in the value-matching condition yields:

$$v(\lambda) = -c\lambda + \lambda\left[v'(\lambda) + c\right]$$

Inserting the obtained solution from (10), this expression becomes:

$$A_1\lambda^{a_1} + \frac{1}{r} = -c\lambda + \lambda\left[A_1 a_1 \lambda^{a_1 - 1} + c\right]$$

which reduces to:

$$A_1 = -\frac{\lambda^{-a_1}}{r(1-a_1)} \tag{14}$$

This expression for A_1 can then be inserted in the value-matching condition in (12) to get:

$$\lambda^{-a_1} - a_1\lambda^{-1} = (1-a_1)(1-rc) \tag{15}$$

which determines λ indirectly.[3] Note that the solution for λ is similar to the one from the deterministic case. The only difference is that $\frac{r}{\alpha}$ is replaced by a_1.

Finally, inserting the expression for A_1 in (10) and multiplying through by Θ, the value function $V(\theta, \Theta)$ can be expressed as:

$$V(\theta,\Theta) = -\frac{\lambda^{-a_1}}{r(1-a_1)}\Theta^{1-a_1}\theta^{a_1} + \frac{\Theta}{r} \tag{16}$$

with a_1 and λ given by (11) and (15), respectively.

The last term in (16) is the present value of the technology in place, whereas the first term expresses the value of the option to invest in new technology. Note that the latter is decreasing in the productivity of the technology in place and increasing in the productivity of the exogenous technology process.

[3] The existence of a unique value of $\lambda > 1$ satisfying (15) can be verified by noting that the left-hand side is monotonically increasing in λ, from $1 - a_1$ to 0, as λ increases from 1 to ∞. The right-hand side, on the other hand, is a constant which is strictly larger than $1 - a_1$ and strictly smaller than 0.

In the Appendix, a more general case is considered where both the productivity of installed technology, Θ, and the productivity of exogenous technology, θ, evolve according to (possibly correlated) geometric Brownian motions. An analytical solution is derived, and it is shown that in the case of zero correlation – which is probably the likely case – the solution corresponds exactly to the solution of a model where only θ (or Θ) evolves stochastically. Hence, whether the uncertainty is modelled in terms of the installed or the exogenous technology – or both – is mainly a technical issue. Focusing on the case with just a single stochastic process, as above, is therefore a convenient simplification.

5 A Comparison with Related Decision Rules

In this Section, the repeated-options rule given by λ in (15) will be compared with two related decision rules in the literature: The simple NPV rule and the single-option rule.

5.1 The Simple NPV Rule

The simple NPV rule says that an investment should be undertaken if the present value of the excess income associated with the investment exceeds the cost. In terms of the above notation, excess income is given by $r^{-1}(\theta - \Theta)$, whereas the cost is $c\theta$. This implies an optimal stopping value, λ, given by:

$$\lambda = \frac{1}{1 - rc} \tag{17}$$

According to the simple NPV criterion, higher cost, c, and higher interest rate, r, both serve to postpone an investment, since they unambiguously decrease the present value of the net returns from the investment. Note also that λ in (17) is independent of uncertainty, σ, since the investment decision is not affected by future dynamics of θ.

5.2 The Single-Option Rule

The core model from Dixit and Pindyck (1994), i.e. the single-option model first analysed by McDonald and Siegel (1986), takes the future dynamics of θ into account, but it does not allow for repeated investments. When an investment is undertaken, the option is "killed". Applying this assumption to the model of the previous Sections, an optimal stopping value, λ, can be shown to be given by the following single-option criterion:

$$\lambda = \frac{1}{\left(1 - \frac{1}{a_1}\right)(1 - rc)} \tag{18}$$

where a_1 is given by (11).[4] Note that the value of λ implied by (18) is always larger than the value implied by (17). The difference reflects the value of waiting. Thus, according to the single-option criterion, the simple NPV criterion dictates too rapid investments.

Using that $\lambda = \theta(\Theta)/\Theta$, the expression in (18) can be rewritten as:

$$\Theta + \frac{r}{a_1}\theta(\Theta)\left(\frac{1}{r} - c\right) = r\theta(\Theta)\left(\frac{1}{r} - c\right) \tag{19}$$

The left-hand side is then the return from holding the option, i.e. from continuing with the installed technology, when the productivity of installed technology is Θ, whereas that of exogenous technology is $\theta(\Theta)$. This return has two components: i) a current income flow of Θ; and ii) an increase in the net present value of stopping. The latter is given by $\frac{r}{a_1}\theta(\Theta)\left(\frac{1}{r} - c\right)$, where $\theta(\Theta)\left(\frac{1}{r} - c\right)$ is the net present value of stopping and obtaining the productivity $\theta(\Theta)$, and r/a_1 is the expected growth rate of this value. Note that r/a_1 increases monotonically from α to r as σ goes from 0 to ∞.[5] Hence, in the deterministic case, the value of stopping grows at the rate of α, the growth rate of productivity of the exogenous technology. However, under uncertainty, the value of stopping is expected to grow at a higher rate. The intuition is the standard one from Dixit and Pindyck (1994), namely that uncertainty has an asymmetric effect on the value of the option, since the agent can react differently to negative and positive realisations of the stochastic variable. In case of a negative shock to θ, he can choose to postpone the investment, thereby mitigating the consequences of a negative shock.

The right-hand side in (19) is the cost of keeping the investment option alive when the productivity of exogenous technology is $\theta(\Theta)$. This cost is given by the interest lost on the value of stopping. Now, for values of θ where the right-hand side is smaller than the left-hand side, i.e. where the cost of keeping the option alive is smaller than the return, it is optimal to keep the option alive and to continue producing Θ with the installed technology. At the stopping line, $\theta(\Theta)$, the two values coincide.

[4] This stopping rule can be found by proceeding exactly as in the case with repeated options with the value of stopping replaced by $\theta\left(\frac{1}{r} - c\right)$.

[5] This follows immediately from (11) and the fact that $\frac{da_1}{d\sigma} < 0$, where the latter can be derived from differentiation of the quadratic equation in (9).

The expression in (19) immediately reveals that the stopping rule, $\lambda = \theta(\Theta)/\Theta$, must be increasing in c. Since $r > r/a_1$, a higher c will decrease the cost of holding the option by more than it decreases the return from holding it. Similarly, λ must be increasing in α and σ, since α and σ will increase r/a_1 and hence imply a higher return from holding the option.[6]

However, the effect of a change in r is more ambiguous. First, a higher interest rate implies a lower net present value of stopping, $\theta(\Theta)\left(\frac{1}{r} - c\right)$, which decreases the cost of holding the option by more than it decreases the return. Secondly, a higher value of r directly increases the current cost of holding the option through the first factor on the right-hand side in (19). Thirdly, the interest rate has an unclear effect on the expected growth rate, r/a_1, of the value of stopping. Hence the aggregate effect of a change in r is ambiguous.

5.3 The Repeated-Options Rule

Now, turn to the repeated-options approach, where the expression for λ in (15) can be rewritten as:[7]

$$\Theta + \frac{r}{a_1}\theta(\Theta)\left(\frac{1}{r} - c\right) = r\theta(\Theta)\left(\frac{1}{r} - c\right) + \frac{1}{a_1}\theta(\Theta)^{1-a_1}\Theta^{a_1} \tag{20}$$

The only difference between this expression and the expression in (19) is the last term on the right-hand side. This term can be interpreted as the additional net cost of keeping the option alive which arises from the existence of future investment options. To see this more clearly, note that the term can alternatively be expressed as:

$$\frac{1}{a_1}\theta(\Theta)^{1-a_1}\Theta^{a_1} = \left(r - \frac{r}{a_1}\right)\left[V(\theta(\Theta), \theta(\Theta)) - \frac{\theta(\Theta)}{r}\right]$$

where the expression in square brackets is the difference in stopping values under the repeated-options approach and the single-option approach. The factor $r - \frac{r}{a_1}$ gives the net cost, i.e. the cost, r, minus the expected return, r/a_1, of holding this extra option value.

It immediately follows from (20) that λ must be smaller under the repeated-options approach than under the single-option approach.[8] Hence the single-option approach dictates too slow investments, when options are actually

[6] Again, the signs of $\frac{da_1}{d\alpha}$ and $\frac{da_1}{d\sigma}$ (and $\frac{da_1}{dr}$) can be derived from differentiation of the quadratic equation in (9).

[7] As in the single-option set-up, the deterministic case corresponds to the situation where $a_1 = \frac{r}{\alpha}$.

[8] To see this, rewrite (20) as: $\Theta - \theta(\Theta)(r - r/a_1)\left(r^{-1} - c\right) = a_1^{-1}\theta(\Theta)^{1-a_1}\Theta^{a_1}$. Given

repeated. By use of (20), it can furthermore be analysed how the optimal stopping rule, λ, is affected by changes in the underlying parameter values.

As for the single-option setup, an increase in c decreases the immediate cost, $r\theta(\Theta)(r^{-1} - c)$, of holding the current option by more than it decreases the immediate return, $\frac{r}{a_1}(\Theta)(r^{-1} - c)$. In addition, it now reduces the value of future options and therefore the cost of holding the additional option value given by the last term in (20). Hence, a higher c will unambiguously reduce the cost of keeping the current option alive by more than it reduces the return. It therefore implies a higher value of λ. More formally, the derivative is given by:

$$\frac{d\lambda}{dc} = \frac{r\left(1 - \frac{1}{a_1}\right)}{\lambda^{-2} - \lambda^{-(1+a_1)}} > 0$$

where the inequality follows from $a_1 > 1$ and $\lambda > 1$.

Compared to the case with a single option, a higher value of α will now have the additional effect of increasing the value of future investment options. This has a positive effect on the last term in (20) and therefore on the cost of keeping the current option alive. This must be compared to the increase in the immediate return from holding the option, which is still a consequence of a higher α. Despite these two countervailing forces, the aggregate effect on λ will still be positive.[9] Formally, the derivative is given by:

$$\frac{d\lambda}{d\alpha} = \frac{\left[rc - 1 + \lambda^{-1} + \lambda^{-a_1}\ln\lambda\right]\frac{da_1}{d\alpha}}{a_1\left(\lambda^{-2} - \lambda^{-(1+a_1)}\right)}$$

where $\frac{da_1}{d\alpha} < 0$.

The effect of an increase in r is even more complex. In addition to the effects from the single-option case, it now decreases the value of future options, thereby making it less costly to keep the current option alive; the last term in (20) becomes smaller. Hence in the repeated-options setup, an increase in r is more likely to cause an increase in λ than in the single-option setup.

Θ, the value of $\theta(\Theta)$ which caues the left-hand side to be zero corresponds to the stopping point from the single-option setup. Since the right-hand side is always positive and the left-hand side is monotonically decreasing in $\theta(\Theta)$, the value of $\theta(\Theta)$ that solves the above equation must be strictly smaller than the value that makes the left-hand side equal to zero.

[9] Unfortunately, it has not been possible to show this formally in terms of the derivative, but careful examination of the parameter space supports the conclusion.

Formally:

$$\frac{d\lambda}{dr} = \frac{-(1-a_1)\,c + \left[rc - 1 + \lambda^{-1} + \lambda^{-a_1}\ln\lambda\right]\frac{da_1}{dr}}{a_1\left(\lambda^{-2} - \lambda^{-(1+a_1)}\right)}$$

where $\frac{da_1}{dr} > 0$.

An increase in σ will, as in the single-option case, increase the immediate return, r/a_1, from keeping the option alive. But in addition, it will now increase the value of future options, thereby increasing the cost of holding the extra option value given by the last term in (20). This is precisely the same effect as with an increase in α. And as in the case of α, the aggregate effect on λ will still be positive:

$$\frac{d\lambda}{d\sigma} = \frac{\left[rc - 1 + \lambda^{-1} + \lambda^{-a_1}\ln\lambda\right]\frac{da_1}{d\sigma}}{a_1\left(\lambda^{-2} - \lambda^{-(1+a_1)}\right)}$$

where $\frac{da_1}{d\sigma} < 0$.[10]

As an attempt to quantify some of these effects, Figures 1 and 2 show examples of how the three decision rules depend on r and α, respectively. From both Figures, it is immediately apparent that the single-option criterion and the repeated-options criterion have rather different implications for optimal stopping behaviour.

In the present example, an increase in the interest rate, r, is seen to increase λ under the repeated-options approach, whereas it decreases λ under the single-option approach. This confirms the intuition from above. The differences between the optimal stopping rules are most pronounced when r is small (close to α). As r is further increased, the future becomes less important, and the repeated-options rule converges to the single-option rule. In the end, as r goes to $\frac{1}{c}$, the stopping rules will converge to infinity.

In Figure 2, an increase in α has a much more pronounced effect on the optimal stopping rule under the single-option approach. This would also be expected from the discussion above.

With respect to σ, Figure 3 shows that a higher σ only marginally influences optimal stopping under the repeated-options approach. This reflects the two countervailing effects of increasing σ that were mentioned above. In the single-option setup, the effect is more visible, since σ unambiguously increases the return from keeping the option alive.

[10] Again, it has not been possible to show formally that $\frac{d\lambda}{d\sigma}$ is always positive. However, numerical results have confirmed that this must be the case.

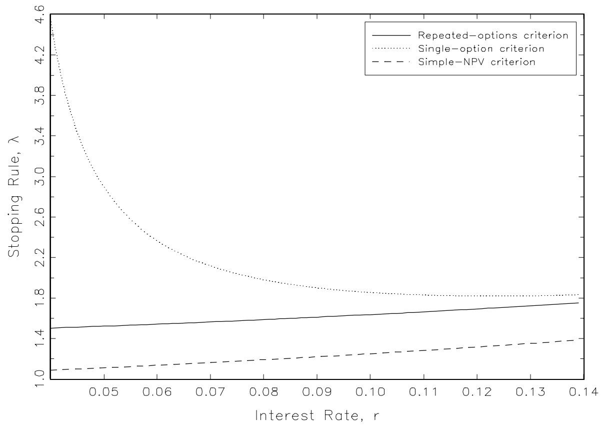

Figure 1: Sensitivity of stopping rules with respect to interest rate, $\alpha = 0.03$, $\sigma = 0.05$, and $c = 2$

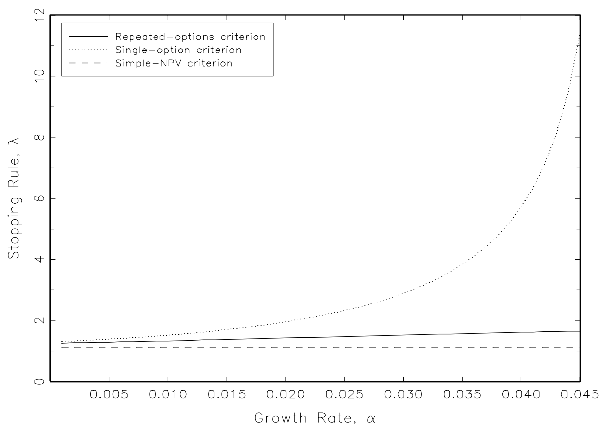

Figure 2: Sensitivity of stopping rules with respect to growth rate, $r = 0.05$, $\sigma = 0.05$, and $c = 2$

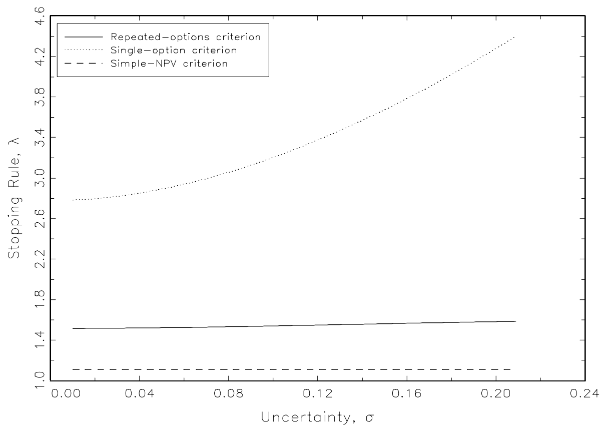

Figure 3: Sensitivity of stopping rules with respect to uncertainty, $\alpha = 0.03$, $r = 0.05$, and $c = 2$

Finally, Figure 4 shows how the normalised value functions vary with changes in σ. Two values of normalised technology have been chosen, $w = 1$ and $w = 1.5$, both inside the continuation region given the chosen parameter values. The effect of uncertainty on the value functions are now more pronounced for the repeated-options case. Though uncertainty does not affect the optimal stopping value, λ, by much, it definitely has positive accumulated effects on wealth in this case.

To sum up, the single-option approach and the repeated-options approach have significantly different implications for optimal investment behaviour – also when it comes to the effects of changes in the underlying parameters. In fact, if investment options are repeated, using the simple NPV criterion with a mark-up to guide investment behaviour seems less of a mistake than using the single-option criterion.

How simple that mark-up might be can be illustrated by looking at Figure 1. For an interest rate of $r = 0.05$, the optimal rule is approximately $\lambda = 1.4$. Actually, using an interest rate of $r = 0.15$ together with the simple NPV-rule will imply a similar value of λ. Since the repeated options solution is relatively insensitive to variations in α and σ, this mark-up, based on a higher interest rate, will be a rather good approximation in many cases. Managers and other decision makers rarely have the time to undertake advanced analyses like the present. The effects of uncertainty and growth must therefore be dealt with in a more ad hoc fashion. The results shown here indicate, that for the case of repeated options, a sensible ad-hoc rule could be a simple mark-up to the interest rate.

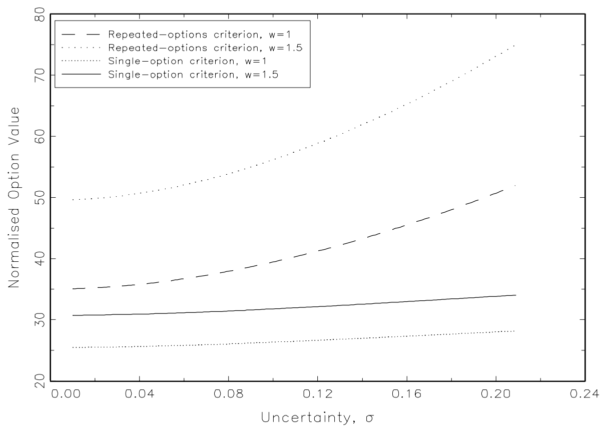

Figure 4: Sensitivity of value functions with respect to uncertainty, $\alpha = 0.03$, $r = 0.05$, and $c = 2$

6 Conclusion

It has been argued in this paper that investment decision problems of the type thoroughly investigated by Dixit and Pindyck (1994) should in many instances be extended to allow for repeated investment options. A model with repeated investment options was solved analytically, where the productivity of exogenous investment options evolved according to a geometric Brownian motion. It was shown that the implications of changes in the structural parameters for optimal investment behaviour were very different when using a repeated-options setup instead of the single-option setup from Dixit and Pindyck (1994). Actually, the simple NPV criterion (with a mark-up) seemed to be a better indicator of optimal investment behaviour than the single-option criterion when investment options are in fact repeated. So perhaps, managers using the simple NPV criterion with a small mark-up, are not performing that bad after all. Or at least not as bad, as one would expect after having studied Dixit and Pindyck (1994) and the related literature.

As stressed in the introduction, the modelling approach in this paper is related to the literature on optimal replacement. The model of this paper, where the productivity of technology follows a geometric Brownian motion, is related to models of Ye (1990) and Mauer and Ott (1995) who analyse replacement problems where maintenance and operation cost of installed equipment follows Ito processes. As shown in the Appendix, analysing uncertainty in terms of the installed technology, as in Ye (1990) and Mauer and Ott (1995), or in terms of the exogenous technology, as in this paper, are actually two sides of the same coin. However, as opposed to Ye (1990) and Mauer and Ott (1995), general analytical solutions are derived in this paper for the case of a geometric Brownian motion, which allows for nice interpretations and a comparison of the optimal decison rule with the one from the now well-established single-option approach.

It should be emphasised that the framework presented in this paper has wide applicability. It can be used as a model of investment behaviour in both firms and in households with home production. Furthermore, in a broader utility sense, the repeated replacement problem is also related to the problem of having the newest car model you can afford, the newest computer or other durables you replace from time to time to boost the utility derived. In a different context, this question has been explored by the literature on optimal consumption and portfolio rules when consumption is (partly) derived from a stock of durables. A benchmark study in this branch of research is Grossman and Laroque (1990) who find that optimal adjustments of the stock of durables are lumpy. Variations over this model include Caballero (1993), Hindy and Huang (1993), and more recently Cuoco and Liu (2000). While the types of control obtained in these studies have similarities to the control of this paper, the advantage of the model and results in the present paper is the simplicity of the derived decision rule and its consequences for optimal behaviour, which makes it practical and easy to apply. Especially, as it is shown in many cases to behave like the well-known simple NPV rule.

One extension of the model in this paper seems obvious: The introduction of credit constraints. This is in many situations a realistic assumption – especially in the case of rural households in less developed countries. When an agent is credit constrained, investments must be financed out of accumulated wealth, and this will add additional aspects to the decision problem – and in general preclude an analytical solution.

Introducing credit constraints in the case of a rural household is not a simple matter, though. When credit is limited, the separability between consumption and production (investment) decisions breaks down. Thus, the investment problem should no longer be analysed separately from the consumption problem; production and consumption decisions become interrelated and must be analysed simultaneously. This is done in Malchow-Møller and Thorsen (2000), where the benchmark model of this paper is used as a model of the production side in an intertemporal household model with credit constraints.

A A General Model

In this appendix, a more general version of the model is solved, where both θ and Θ evolve according to stochastic processes. Specifically, let:

$$\begin{aligned} d\theta &= \alpha_1 \theta dt + \sigma_1 \theta dz_1 \\ d\Theta &= \alpha_2 \Theta dt + \sigma_2 \Theta dz_2 \end{aligned}$$

where α_1, α_2, σ_1, and σ_2 are the parameters of the respective processes, and dz_1 and dz_2 are increments of Wiener processes with $E(dz_1 dz_2) = \rho$. It is assumed that $r > \alpha_1$. The value function can then be expressed as:

$$\begin{aligned} V(\theta, \Theta) &= \max \{ \Theta dt + (1 + rdt)^{-1} E[V(\theta + d\theta, \Theta + d\Theta) | \theta, \Theta], \\ &\quad \theta dt - c\theta + (1 + rdt)^{-1} E[V(\theta + d\theta, \theta + d\Theta) | \theta, \Theta = \theta] \} \end{aligned}$$

In the continuation region, the following partial differential equation applies:

$$rV = \Theta + \theta \alpha_1 V'_\theta + \frac{1}{2} \theta^2 \sigma_1^2 V''_{\theta\theta} + \Theta \alpha_2 V'_\Theta + \frac{1}{2} \Theta^2 \sigma_2^2 V''_{\Theta\Theta} + \rho \theta \Theta \alpha_1 \alpha_2 V''_{\theta\Theta} \quad (21)$$

Again, the homogeneity of $V(\theta, \Theta)$ can be used to define a normalised value function:

$$v(w) = \Theta^{-1} V(\theta, \Theta)$$

where $w = \theta / \Theta$, and:

$$\begin{aligned} &V'_\theta = v' \quad , \quad V'_\Theta = v - wv' \\ &V''_{\theta\theta} = \Theta^{-1} v'' \quad , \quad V''_{\Theta\Theta} = \Theta^{-1} w^2 v'' \quad , \quad V''_{\theta\Theta} = -\Theta^{-1} wv'' \end{aligned}$$

Substituting for V and its derivatives in the partial differential equation in (21), it can be written as the following ordinary differential equation:

$$v(r - \alpha_2) + wv'(\alpha_2 - \alpha_1) + \frac{1}{2} w^2 v'' \left(2\rho\alpha_1\alpha_2 - \sigma_1^2 - \sigma_2^2\right) = 1 \quad (22)$$

which has the general solution:

$$v = A_1 w^{a_1} + A_2 w^{a_2} + K$$

Substituting this back into the differential equation in (22) yields $K = (r - \alpha_2)^{-1}$, and a_1 and a_2 as the roots of:

$$a^2 \left(\rho\alpha_1\alpha_2 - \frac{\sigma_2^2}{2} - \frac{\sigma_2^2}{2} \right) + a \left(\alpha_2 - \alpha_1 - \rho\alpha_1\alpha_2 + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} \right) + r - \alpha_2 = 0$$

Hence:

$$a_1 = \frac{1}{2} - \frac{\alpha_1 - \alpha_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\alpha_1\alpha_2} + \sqrt{\left(\frac{1}{2} - \frac{\alpha_1 - \alpha_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\alpha_1\alpha_2}\right)^2 + \frac{2(r - \alpha_2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\alpha_1\alpha_2}} > 1$$

and:

$$a_2 = \frac{1}{2} - \frac{\alpha_1 - \alpha_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\alpha_1\alpha_2} - \sqrt{\left(\frac{1}{2} + \frac{\alpha_1 - \alpha_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\alpha_1\alpha_2}\right)^2 + \frac{2(r - \alpha_1)}{\sigma_1^2 + \sigma_2^2 - 2\rho\alpha_1\alpha_2}} < 0$$

A_2 must therefore equal zero, otherwise V would approach infinity as w becomes small. The value-matching and smooth-pasting conditions are of the same form as in the paper, implying that A_1 and λ are given by:

$$A_1 = -\frac{\lambda^{-a_1}}{(r - \alpha_2)(1 - a_1)}$$

and:

$$\lambda^{-a_1} - a_1\lambda^{-1} = (1 - a_1)(1 - (r - \alpha_2)c)$$

Note that if $\rho = 0$, $\alpha_2 = 0$, and $\sigma_2^2 = 0$, the solution reduces to the one from the paper. Note also that when $\rho = 0$, the model with two stochastic processes above could be transformed into a model with a single stochastic process of the type in the paper, using the parameters $\alpha = \alpha_1 - \alpha_2$ and $\sigma = \sigma_1^2 + \sigma_2^2$, together with the modified interest rate $r - \alpha_2$. This justifies the approach taken in the paper.

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