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Government Debt and Capital Accumulation in the Blanchard-Cass-Yaari OLG Model

Bo Sandemann Rasmussen*

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Abstract

It is shown that although government debt in principle has an ambiguous effect on the steady state capital stock in an OLG model of the Blanchard-Cass-Yaari variety, once stability of the steady state equilibrium is imposed there is an unambiguous negative relation between the level of government debt and the capital stock.

Keywords: OLG model, government debt, stability, capital accumulation.

JEL: E13, H63.

1. Introduction

The impact of government debt on the economy is an important topic once Ricardian equivalence fails to hold, e.g. due to households having finite lives. The Blanchard-Cass-Yaari model of overlapping generations where agents continuously face a risk of dying has become a popular model for analyses of the impact of public debt (see e.g. Gertler (1999) for a recent contribution that extends the Blanchard-Cass-Yaari model to include stochastic retirement). In that model public debt matters since it redistributes wealth across generations. With finite horizons in the private sector an intertemporal reallocation of taxes that allows for a permanent increase in the level of government debt (but satisfies the government's intertemporal solvency condition) will increase the stock of human wealth of the households currently alive and reduce the stock of human wealth of households born into the future as they must pay higher taxes to cover the increased interest payments on the public debt. These reallocations of wealth across generations will

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affect saving behaviour and hence capital accumulation, and a simple, and policy relevant, question is in what direction the aggregate capital stock is affected. In Blanchard and Fischer (1989, ch. 3) this is analyzed rigorously and the conclusion is that the effect is generally ambiguous, but if labour income exceeds the tax payments there is unambiguously a negative effect on the steady state capital stock of an increase in government debt. The purpose of the present paper is to show that by imposing stability of the steady state equilibrium the ambiguity can be removed, implying that an unambiguous negative relation between the steady state capital stock and the level of government debt exists.

2. The Model

The model follows closely Blanchard (1985) and Blanchard and Fischer (1989, ch. 3). At each instant a cohort of size $p > 0$ is born, and by assuming that households face a constant death rate equal to the birth rate, p, the population is constant at a size of one. Given the probability of dying there is uncertainty about individual lifetime, but due to each cohort being "big" (the size p is a normalization) there is no aggregate uncertainty: Each cohort declines deterministically through time such that a cohort born at time s will be of size $pe^{-p(t-s)}$ at time t. The presence of individual uncertainty and aggregate certainty implies that any individual income uncertainty (due to uncertain lifetimes) can be removed through an insurance market. Competitive insurance companies offer "insurance" of the form where each individual receives the flow $pv(t)$, where $v(t)$ is the level of nonhuman wealth of the individual, when alive in exchange of paying the full amount $v(t)$ to the insurance companies when they die. The insurance companies balance their books and the individuals avoid either leaving unintended bequests or running out of wealth before dying.

The von Neumann-Morgenstern expected utility function is

$$EU(t) = \int_t^\infty \ln(c(z))\, e^{-(\theta+p)(z-t)} dz, \tag{2.1}$$

where $\theta > 0$ is the instantaneous rate of time preference, $c(z)$ is consumption at time z and the instantaneous utility function is assumed to be logaritmic. The flow budget constraint reads

$$\dot{v}(z) = (r(z) + p)\, v(z) + y(z) - c(z) - t(z), \tag{2.2}$$

$r(z)$ being the interest rate, $y(z)$ labour income and $t(z)$ (lump sum) taxes. Finally, to avoid a trivial solution to the optimal consumption choice problem a No-Ponzi-Game condition is imposed,

$$\lim_{z\to\infty} v(z) e^{-\int_t^z (r(\nu)+p)d\nu} = 0. \tag{2.3}$$

Using the maximum principle the Euler equation becomes

$$\dot{c}(z) = [r(z) - \theta]\, c(z), \tag{2.4}$$

which has the solution

$$c(z) = c(t)e^{\int_t^z (r(\nu)-\theta)d\nu}. \tag{2.5}$$

Calculating the present value of consumption yields,

$$\int_t^\infty c(z)e^{-\int_t^z (r(\nu)+p)d\nu}dz = \frac{c(t)}{\theta + p} = v(t) + h(t), \tag{2.6}$$

where the second part of the equation follows from the present value budget constraint, $h(t)$ being human wealth defined as the present value of labour income net of taxes

$$h(t) = \int_t^\infty [y(z) - t(z)]\, e^{-\int_t^z (r(\nu)+p)d\nu}dz. \tag{2.7}$$

Thus, the individual consumption function is

$$c(t) = (\theta + p)\,(v(t) + h(t))\,. \tag{2.8}$$

To obtain aggregate consumption, $C(t)$, we just add consumption of all individuals alive at time t (now $c(s,t)$ denotes consumption at time t of an individual born at time s and similarly for other individual variables):

$$C(t) = \int_{-\infty}^t c(s,t)pe^{-p(t-s)}ds, \tag{2.9}$$

where $pe^{-p(t-s)}$ is the size at t of the cohort born at time s. Inserting the individual consumption functions and using the definitions of aggregate human and nonhuman wealth, $H(t)$ and $V(t)$ respectively,

$$H(t) = \int_{-\infty}^t h(s,t)pe^{-p(t-s)}ds, \tag{2.10}$$

$$V(t) = \int_{-\infty}^t v(s,t)pe^{-p(t-s)}ds, \tag{2.11}$$

we get

$$C(t) = (\theta + p)\,(V(t) + H(t))\,. \tag{2.12}$$

The dynamics of aggregate consumption follows from the dynamics of human and nonhuman wealth:

$$\dot{C}(t) = (\theta + p)\left(\dot{V}(t) + \dot{H}(t)\right). \tag{2.13}$$

Aggregate human wealth is equal to

$$H(t) = \int_t^\infty [Y(z) - T(z)]\, e^{-\int_t^z (r(v)+p)dv} dz,$$

where $Y(z)$ and $T(z)$ are aggregate labour income and lump sum taxes at time z, respectively. It is here implicitly assumed that individual labour income is distributed independently of age. Differentiating $H(t)$ with respect to time leads to

$$\dot{H}(t) = (r(t) + p)\, H(t) - Y(t) + T(t). \tag{2.14}$$

Aggregate nonhuman wealth consists of physical capital, $K(t)$, and government bonds, $B(t)$,

$$V(t) = K(t) + B(t), \tag{2.15}$$

such that the dynamics of nonhuman wealth follows from the dynamics of physical capital and government debt.

The net production function, $F(K(t))$, satisfies the usual neoclassical assumptions (including constant returns to scale and the Inada conditions). The demand for capital by profit maximizing firms becomes

$$F^{'}(K(t)) = r(t), \tag{2.16}$$

while labour income amounts to

$$Y(t) = F(K(t)) - r(t)K(t),$$

implying zero pure profits.

The dynamics of the capital stock follows from the aggregate resource constraint

$$\dot{K}(t) = F(Kt)) - C(t) - G(t) = Y(t) + r(t)K(t) - C(t) - G(t), \tag{2.17}$$

where $G(t)$ is government consumption. The government's flow budget constraint reads[1]

$$\dot{B}(t) = r(t)B(t) + G(t) - T(t). \tag{2.18}$$

Thus, using equations 2.17 and 2.18 the dynamics of nonhuman wealth become

$$\dot{V}(t) = r(t)V(t) + Y(t) - C(t) - T(t). \tag{2.19}$$

[1] As usual, a No-Ponzi-Game condition is imposed on the government

$$\lim_{z\to\infty} B(z)e^{-\int_t^z r(\nu)d\nu} = 0,$$

such that the flow budget constraint can be integrated into a present value budget constraint.

Using equations 2.14 and 2.19 aggregate consumption dynamics can be expressed as

$$\dot{C}(t) = (r(t) - \theta)\, C(t) - p\,(\theta + p)\, V(t). \tag{2.20}$$

Thus, we can state the aggregate dynamics of the model (time indices are left out from now on):

$$\begin{aligned} \dot{C} &= \left(F^{'}(K) - \theta\right) C - p\,(\theta + p)\,(B + K) & (2.21) \\ \dot{K} &= F(K) - C - G & (2.22) \\ \dot{B} &= F^{'}(K)B + G - T. & (2.23) \end{aligned}$$

3. Steady State Equilibrium

In a steady state equilibrium the levels of the endogenous variables must be constant, $\dot{C} = \dot{K} = \dot{B} = 0$ (this basically follows from the Inada conditions ruling out endogenous growth). Denoting steady state equilibrium levels by an asterisk, "*", the steady state equilibrium is characterized by

$$\begin{aligned} C^{*} &= \frac{p\,(\theta + p)\,(B^{*} + K^{*})}{F^{'}(K^{*}) - \theta} = F(K^{*}) - G & (3.1) \\ B^{*} &= \frac{G - T}{F^{'}(K^{*})} & (3.2) \\ r^{*} &= F^{'}(K^{*}). & (3.3) \end{aligned}$$

(Notice that $F^{'}(K^{*}) = r^{*} > \theta$ for $C^{*} > 0$ implying the well-known result that the steady state capital stock in the Blanchard-Cass-Yaari model is below the level implied by the "modified Golden Rule" stating that $F^{'}(K^{*}) = \theta$.) By the Inada condition

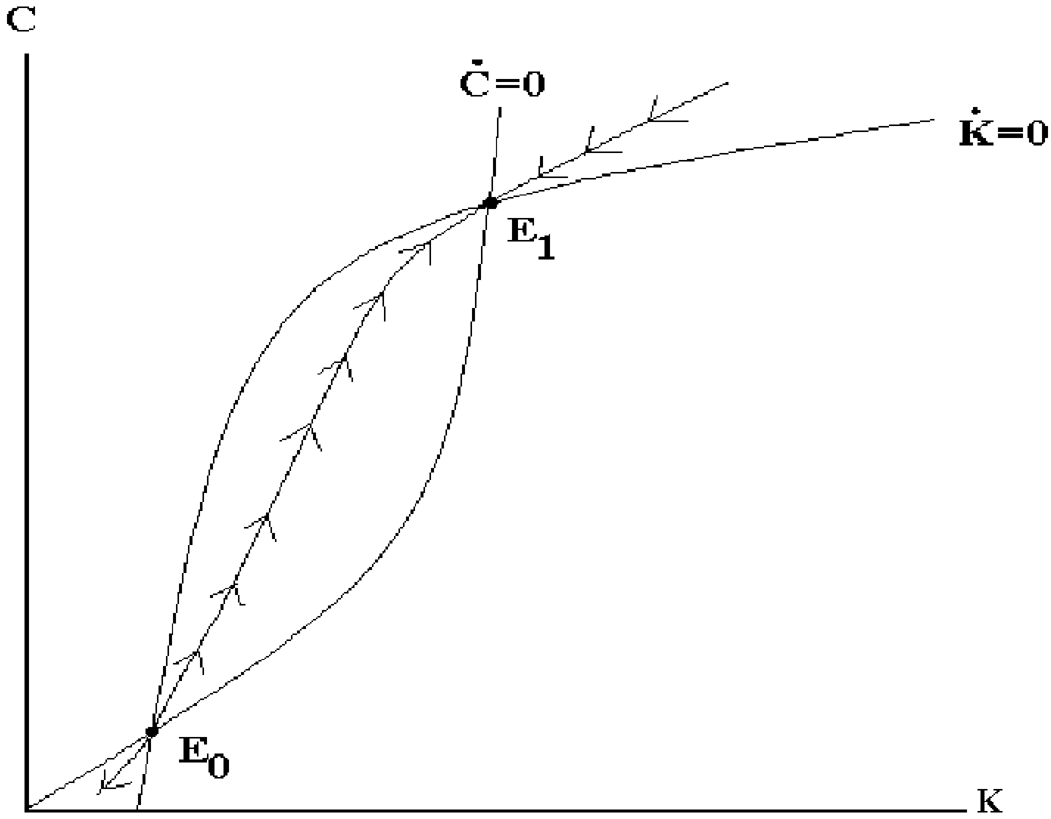
$$\lim_{K \to 0} F'(K) = \infty,$$

the stationary trajectory for C starts at $(C, K) = (0, 0)$. Assuming convexity of $\dot{C} = 0$,[2] the concavity of $\dot{K} = 0$ implies that there exists either zero, one or two steady state equilibria. For a sufficiently large value of G there exists no steady state equilibrium, while in the limiting case where the stationary trajectories for C

[2] The stationary trajectory for C need not be convex in (C, K)-space since the sign of $\left.\frac{d^2C}{dK^2}\right|_{\dot{C}=0}$ depends on the third derivative of the net production function. We will assume convexity holds, for simplicity (which will be the case if e.g. the net production function is Cobb-Douglas), but our results are easily extended to cover non-convexity of the stationary trajectory for C.

and K are tangential to each other a unique steady state equilibrium exists. Both of these cases are disregarded in the following by assuming that G is sufficiently small. Hence, we consider the general case where two steady state equilibria exist, see Figure 1.

Figure 1: Stability of Steady State Equilibria



3.1. Local Stability

To consider the stability of the steady state equilibria we linearize the dynamic equations around a steady state equilibrium:[3]

$$\begin{bmatrix} \dot{C} \\ \dot{K} \end{bmatrix} = \begin{bmatrix} r^* - \theta, & F''(K^*)C^* - p(\theta + p) \\ -1, & r^* \end{bmatrix} \begin{bmatrix} C - C^* \\ K - K^* \end{bmatrix} \tag{3.4}$$

[3] In the policy experiment the level of government debt changes discretely whereafter taxes are set such that the are no subsequent changes in debt. Hence, we can consider the dynamic system as a two-variable system in (C, K). We can then from the government budget constraint determine the level of taxes needed to hold the level of debt constant.

The eigenvalues are determined by

$$\begin{vmatrix} r^* - \theta - \lambda, & F^{''}(K^*)C^* - p\,(\theta + p) \\ -1, & r^* - \lambda \end{vmatrix} = 0, \tag{3.5}$$

or

$$\lambda^2 - (2r^* - \theta)\,\lambda + F^{''}(K^*)C^* - p\,(\theta + p) + r^*\,(r^* - \theta) = 0. \tag{3.6}$$

The solution to this quadratic equation is

$$\lambda = \frac{2r^* - \theta \pm \left[(2r^* - \theta)^2 - 4\left(F^{''}(K^*)C^* - p\,(\theta + p) + r^*\,(r^* - \theta)\right)\right]^{\frac{1}{2}}}{2}. \tag{3.7}$$

Since the capital stock is a predetermined variable while consumption is a jump variable we require that the two roots be of opposite sign for the steady state to be saddle path stable, while the equilibrium will be unstable if both roots are positive. The result is stated in the following lemma.

Lemma 1. *(Local) Stability: The steady state equilibrium is locally stable if*

$$\frac{p\,(\theta + p) - F^{''}(K^*)C^*}{r^* - \theta} > r^*.$$

Proof. *It follows directly from 3.7 that the two roots will be of opposite sign provided* $\frac{p(\theta+p)-F^{''}(K^*)C^*}{r^*-\theta} > r^*$. ■

Calculating the slopes of the stationary trajectories for C and K,

$$\left.\frac{dC}{dK}\right|_{\dot{C}=0} = \frac{p(p+\theta) - F^{''}(K^*)C^*}{r^* - \theta} > 0 \tag{3.8}$$

$$\left.\frac{dC}{dK}\right|_{\dot{K}=0} = r^* > 0, \tag{3.9}$$

it follows that the stability condition can be stated in terms of the relative slopes of the stationary trajectories. Thus,

$$\left.\frac{dC}{dK}\right|_{\dot{C}=0} > \left.\frac{dC}{dK}\right|_{\dot{K}=0} \Rightarrow \text{local stability.} \tag{3.10}$$

This can be illustrated by considering the dynamics of the economy in Figure 1 where the steady state equilibrium E_0 is unstable while E_1 is stable.

4. Government Debt and Capital Accumulation

Consider now an increase in steady state government debt brought about by a temporary decrease in taxation followed by an increase in taxation that keeps the level government debt constant at every subsequent instant. Calculating the effect on the steady state capital stock from equation 3.1 yields (after some manipulations)

$$\frac{dK^*}{dB^*} = \frac{p\,(p+\theta)}{F''(K^*)C^* - (r^*-\theta)(Y^*-T)/\,(B^*+K^*)} \tag{4.1}$$

such that $Y^* > T$ is a sufficient condition for $\frac{dK^*}{dB^*} < 0$, as argued by Blanchard and Fischer (1989, ch. 3). However, since total household income is $(r^*\,(B^*+K^*)+Y^*)$ there is no reason why $Y^* > T$ should hold, implying that we seemingly cannot sign $\frac{dK^*}{dB^*}$. Moreover, since $Y^* < T$ (by a small amount) could be consistent with $\frac{dK^*}{dB^*} < 0$, there is hardly any intuition behind this condition. Fortunately, there is a much better argument for ruling out a positive relation between steady state government debt and the capital stock based on a stability argument.

Given the multiple equilibrium property of the steady state equilibria it seems necessary to take the stability properties of the various equilibria into account. It tuns out that imposing stability of the steady state equilibrium is sufficient to rule out the possibility of a positive relationship between government debt and capital accumulation.

Proposition 1. *A necessary and sufficient condition for* $\frac{dK^*}{dB^*} < 0$, *is that*

$$\left.\frac{dC}{dK}\right|_{\dot{C}=0} > \left.\frac{dC}{dK}\right|_{\dot{K}=0}. \tag{4.2}$$

Proof. *Using the expressions for the slopes of the stationary trajectories given in equations 3.8 and 3.9, the steady state effect on the capital stock of an increase in government debt can be written as*

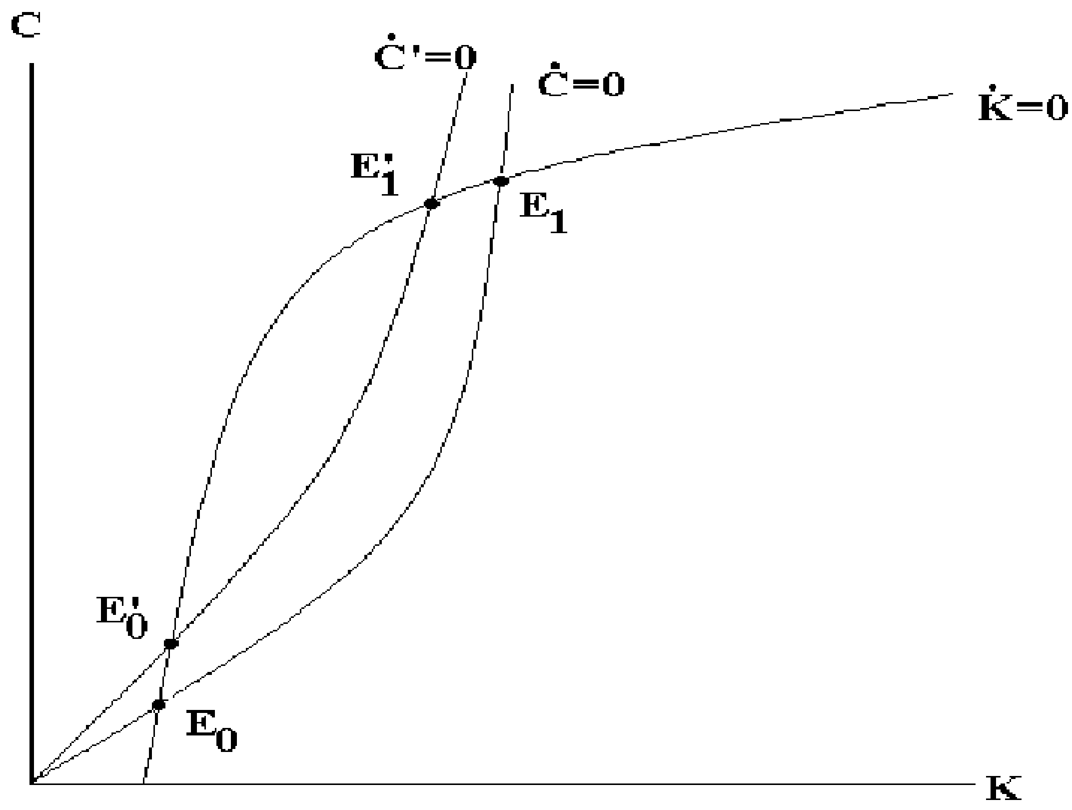
$$\frac{dK^*}{dB^*} = -\frac{(r^*-\theta)\,p\,(p+\theta)}{\left.\frac{dC}{dK}\right|_{\dot{C}=0} - \left.\frac{dC}{dK}\right|_{\dot{K}=0}}, \tag{4.3}$$

implying that a necessary and sufficient condition for a negative effect on the capital stock of an increase in government debt is that the stationary trajectory for consumption is steeper than the stationary trajectory for capital. ■

Thus, since condition 4.2 is exactly what characterizes a (locally) stable steady state equilibrium, an unambiguous negative steady state relation exists if stability of the steady state equilibrium is imposed. Hence, we do not need to impose an arbitrary condition like $Y^* > T$ for the steady state effect to be uniquely determined.

As an illustration, consider Figure 2, where E_0 is the initial unstable steady state equilibrium while E_1 is the initial stable steady state equilibrium.

Figure 2. Steady State Effects of Increasing Government Debt



A rise in government debt rotates the $\dot{C} = 0$ schedule upwards (to $\dot{C}' = 0$) while leaving the $\dot{K} = 0$ schedule unchanged. Then, starting from the unstable equilibrium the effect on the capital stock is seemingly positive, from E_0 to E_0', while the effect is negative when starting from the stable equilibrium, E_1 to E_1'. Obviously, disregarding the unstable equilibrium removes the ambiguity of the effect on the capital stock from an increase in government debt.

5. Conclusion

It has been shown that the sign of the effect on the capital stock of an increase in the level of government debt can be uniquely determined by imposing stability of the steady state equilibrium. Given stability there exists an unambiguously negative relation between the steady state capital stock and the level of government debt. No assumptions about the relative size of labour income and lump sum taxes are needed to determine the sign of this relation.

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